A decision making methodology based on the weighted correlation coefficient in weighted extended hesitant fuzzy environments

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Abstract. Correlation is an important index in decision-making. In weighted extended hesitant fuzzy sets (WEHFSs) environment, researchers have only defined a class of correlation coefficients between WEHFSs with values in the unit interval [0,1]. This is not ideal because it does not extend the classical correlation coefficient in the case of classical sets. In fact, the negative values of the interval [-1,1] are ignored, and such a neglectfulness leads to unreasonable results in decision making. In other words, the existing definitions are unconvincing and lack consistency, which hinder their application potentials. This article addresses this issue by introducing a new class of weighted correlation coefficients of WEHFSs with values in the interval [-1,1]. Three decision making methodologies based on the weighted correlation coefficients of WEHFSs are compared with the existing methodologies based on their respective correlation coefficients in the unit interval [0,1]. The comparative analysis shows both the efficiency and effectiveness of the new correlation index.

Keywords: Weighted correlation; Weighted correlation coefficient; Weighted extended hesitant fuzzy set; Decision making.

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1 Introduction

Decision making is a pervasive task in people’s daily routine. The aim in decision making is the selection of the most preferable alternative among a number of given alternatives using the analysis of information provided by a board of decision makers. Depending on the type of information uncertainty in decision-making procedures, different decision-based techniques have been investigated on the basis of hesitant fuzzy sets (HFSs) [6, 15], interval-valued hesitant fuzzy sets (IVHFSs) [7], dual hesitant fuzzy sets (DHFSs) [9], higher order hesitant fuzzy sets (HOHFSs) [8], hesitant fuzzy linguistic term sets (HFLTSs) [10], etc.

A frequently used index in decision making procedures is the correlation coefficient [4, 24, 28], which describes the behaviour of moving together two variables in a linear fashion. Indeed, the index of correlation coefficient measures the degree of correlation of two observed datasets. More specifically, in the case where all the information provided by decision makers is at hand, we then are able to rank all the alternatives using correlation coefficients between each alternative and an ideal alternative. However, the concept of correlation coefficient is widely used in a number of fields such as biomedical [25], pattern recognition [5], artificial intelligence [41], earthquake engineering [1], etc. The most common classic statistical form of correlation coefficient is that of Karl Pearson, which was considered for crisp observed datasets. However, we are not always able to obtain the crisp values of variables in real-world applications due to the presence of ambiguity and uncertainty. This motivated the study of the correlation coefficient of fuzzy observed datasets. In this regard, up to now, different forms of correlation have been reported in the fuzzy literature, among them are the fuzzy and the intuitionistic fuzzy versions of correlation coefficient [18, 27]. Furthermore, in recent years, numerous contributions have appeared on the subject of correlation coefficient in hesitant fuzzy environment. Xu and Xia [36] proposed a number of correlation coefficients for HFSs from the view point of information energy, which was subsequently extended by Chen et al. [2] correlation coefficients for IVHFSs. Later, Liao et al. [20] extended the notion of correlation coefficient to the case of HFLTSs, while Wang et al. [33], Ye [40], and Tyagi et al. [31] extended this concept to the case DHFSs; Farhadinia [9] to interval-valued DHFSs; and Yang et al. [39] to hesitant multiplicative set. In addition, Guan et al. [16] and Sun et al. [30] developed correlation coefficients based on the notion of mean, variance and hesitancy degree of HFSs.
There are mainly two categories of correlation coefficients proposed so far: based on the statistical viewpoint, and on the basis of information-energy. The values of the latter class of correlation coefficients are within the interval $[0, 1]$, while the values of the former class of correlation coefficients lie in the interval $[-1, 1]$. Clearly, the information-energy class of correlation coefficients are not able to represent the positive or negative relationship between different sets which lacks consistency with classical correlation hinders wider application potentials.

As an extension of the HFS concept, Zhu and Xu [43] proposed the concept of extended hesitant fuzzy set (EHFS) as a tool to avoid loss of information. In fact, such a consideration enables us to present information represented by the decision makers using possible value-groups. However, this definition of EHFS suffers from some drawbacks which entail appropriate revisiting before it can be used. In a fresh study conducted by Farhadinia and Herrera-Viedma [11], the concept of EHFS proposed by Zhu and Xu was revisited using the Cartesian product of HFEs. Farhadinia and Xu [14] have recently dealt with a new aspect of emergency event that used instead of usual aggregation procedure, a new fusion technique based on the modified version of EHFS. The key element of this technique is that it keeps all possible amount of expert’s information better than the existing fusion technique of HFS.

Up to now, few studies exist on the correlation coefficients for EHFSs. Lu and Liang [21] defined two classes of correlation coefficients for EHFSs, one of which consider the extended hesitant fuzzy elements (EHFEs) length. The correlation coefficient concept has been recently generalized by Lu and Liang [21] to EHFSs, although they defined this concept positively, that is, with values in the interval $[0, 1]$, while the negative values of the classical interval $[-1, 1]$ are ignored. As mentioned above, such neglectfulness leads to unreasonable results in decision making. Consequently, a new definition of correlation coefficient concept for EHFSs is still in need. This is the main motivation for developing a new class of EHFS correlation coefficient with values in the interval $[-1, 1]$. Its superiority compared to the existing correlation coefficients defined on the interval $[0, 1]$ will be demonstrated with numerical experiments presented in Section 4.

The remainder of this paper is set out as the following. Section 2 presents briefly some basic knowledge on the EHFS concept. Section 3, the main contribution of the paper, deals with the correlation coefficients for
EHFSs, and then the weighted correlation coefficients for WEHFSs in \([-1, 1]\] are presented. Section 4 is divided into three subsections with three decision making methodologies, one based on EHFEs and the others on the length-normalized HFEs which are indeed EHFEs, to allow the comparison between the proposed correlation coefficients and the existing ones. Finally, the paper ends with some conclusions remarks in Section 5.

2 Preliminaries

In order to make the paper self-contained, we present below the seminal concept of hesitant fuzzy set (HFS). This section continues with reviewing Farhadinia and Herrera-Viedma’s [11] modification version of the extended hesitant fuzzy set (EHFS) concept initially proposed by Zhu and Xu at [43]. Then, a number of requirements are provided which are needed in the subsequent sections.

**Definition 2.1.** [32] Let \(X\) be the referential set. Then, a hesitant fuzzy set (HFS) on \(X\) is represented as

\[A = \{\langle x, h_A(x) \rangle : x \in X, \ h_A(x) \subseteq [0, 1] \} \]

The term \(h_A(x)\) is known as the hesitant fuzzy element (HFE) of \(A\), and it returns all possible membership degrees of \(x \in X\) in the set \(A\).

Hereafter, without loss of generality, we consider \(h_1 = \{h_1^{(\delta_1)} \mid \delta_1 = 1, \ldots, l_{h_1}\}\) and \(h_2 = \{h_2^{(\delta_2)} \mid \delta_2 = 1, \ldots, l_{h_2}\}\) as two arbitrary HFEs of lengths \(l_{h_1}\) and \(l_{h_2}\), respectively, which are not necessarily related to the same element \(x \in X\) and the same HFS \(A\). Further, the notation \(h^{(\delta)}\) stands for the \(\delta\)-th element in the HFE \(h\). Keeping these notations in mind, we present below the definition of some algebraic operations on HFEs (see [35]):

- **Addition:** \(h_1 \oplus h_2 = \bigcup_{h_1^{(\delta_1)} \in h_1, h_2^{(\delta_2)} \in h_2} \{h_1^{(\delta_1)} + h_2^{(\delta_2)} - h_1^{(\delta_1)} h_1^{(\delta_2)}\};\)
Multiplication: \( h_1 \otimes h_2 = \bigcup_{h_1^{(d_1)} \in h_1, h_2^{(d_2)} \in h_2} \{ h_1^{(d_1)} h_2^{(d_2)} \}; \)

Multiplication by scalar: \( \lambda h_1 = \bigcup_{h_1^{(d_1)} \in h_1} \{ 1 - (1 - h_1^{(d_1)})^\lambda \}, \lambda > 0; \)

Power: \( h_1^\lambda = \bigcup_{h_1^{(d_1)} \in h_1} \{ (h_1^{(d_1)})^\lambda \}, \lambda > 0. \)

What is essential to be considered in the above computation procedures is the length unification of all taken HFEs. In most situations, for any two HFEs \( h_1 \) and \( h_2 \), we observe that \( l_{h_1} \neq l_{h_2} \). In order to compare \( h_1 \) and \( h_2 \) correctly, the shorter HFE is extended so that both HFEs have the same length [34]. This unification process is conducted in three manners (see, for example, [15]): if \( l = \max\{l_{h_1}, l_{h_2}\} \), then (i) the minimum value will be repeated in the case of a pessimistic scenario; (ii) the maximum value will be repeated in the case of an optimistic scenario; and (iii) a linear combination of maximum value and minimum value in the given HFE will be repeated according to the decision makers’ risk preferences.

Now, we present the revised form of extended hesitant fuzzy set (EHFS), which was initially proposed by Zhu and Xu at [43]. Afterwards, Farhadinia and Herrera-Viedma [11] proposed a revised version based on the Cartesian product of HFEs in which an EHF is considered as the n-tuples that indicate simultaneously the opinion of "m" decision makers.

**Definition 2.2.** [11] Let \( X \) be the referential set. An extended hesitant fuzzy set (EHFS) is represented in the following form:

\[
\mathcal{H} = \{ (x, h(x)) \mid x \in X \} = \{ (x, \bigcup_{(\gamma_1(x), \ldots, \gamma_m(x)) \in h(x)} \{ \gamma_1(x), \ldots, \gamma_m(x) \}) \mid x \in X \},
\]

where

\[
h = \bigcup_{(\gamma_1, \ldots, \gamma_m) \in \mathcal{H}} \{ \gamma_1, \ldots, \gamma_m \},
\]

denotes an extended HFE (EHFE) containing some m-tuples which stand for the opinion of m decision makers, simultaneously.
For any two EHFEs $\mathbf{h}_1 = \{h_1^{(\delta_1)} := (\gamma_1^{(\delta_1)}, ..., \gamma_m^{(\delta_1)}) \mid \delta_1 = 1, ..., l_{h_1}\}$ and $\mathbf{h}_2 = \{h_2^{(\delta_2)} := (\gamma_1^{(\delta_2)}, ..., \gamma_m^{(\delta_2)}) \mid \delta_2 = 1, ..., l_{h_2}\}$ of lengths $l_{h_1}$ and $l_{h_2}$, respectively, the following operations are defined:

- **Addition:**

$$\mathbf{h}_1 \oplus \mathbf{h}_2 = \bigcup_{h_1^{(\delta_1)} \in \mathbf{h}_1, h_2^{(\delta_2)} \in \mathbf{h}_2} \{h_1^{(\delta_1)} + h_2^{(\delta_2)} - h_1^{(\delta_2)}h_2^{(\delta_1)}\}$$

$$= \bigcup_{(\gamma_1^{(\delta_1)}, ..., \gamma_m^{(\delta_1)}) \in \mathbf{h}_1, (\gamma_1^{(\delta_2)}, ..., \gamma_m^{(\delta_2)}) \in \mathbf{h}_2} \{(\gamma_1^{(\delta_1)} + \gamma_1^{(\delta_2)} - \gamma_1^{(\delta_1)}\gamma_1^{(\delta_2)}), ..., \gamma_m^{(\delta_1)} + \gamma_m^{(\delta_2)} - \gamma_m^{(\delta_1)}\gamma_m^{(\delta_2)}\};$$

(3)

- **Multiplication:**

$$\mathbf{h}_1 \otimes \mathbf{h}_2 = \bigcup_{h_1^{(\delta_1)} \in \mathbf{h}_1, h_2^{(\delta_2)} \in \mathbf{h}_2} \{h_1^{(\delta_1)}h_2^{(\delta_2)}\}$$

$$= \bigcup_{(\gamma_1^{(\delta_1)}, ..., \gamma_m^{(\delta_1)}) \in \mathbf{h}_1, (\gamma_1^{(\delta_2)}, ..., \gamma_m^{(\delta_2)}) \in \mathbf{h}_2} \{(\gamma_1^{(\delta_1)}\gamma_1^{(\delta_2)}), ..., \gamma_m^{(\delta_1)}\gamma_m^{(\delta_2)}\};$$

(4)

- **Multiplication by scalar:**

$$\lambda \mathbf{h}_1 = \bigcup_{h_1^{(\delta_1)} \in \mathbf{h}_1} \{1 - (1 - h_1^{(\delta_1)})^\lambda\}$$

$$= \bigcup_{(\gamma_1^{(\delta_1)}, ..., \gamma_m^{(\delta_1)}) \in \mathbf{h}_1} \{(1 - (1 - \gamma_1^{(\delta_1)}), ..., 1 - \gamma_m^{(\delta_1)}))^\lambda}\}, \lambda > 0;$$

(5)

- **Power:**

$$\mathbf{h}_1^\lambda = \bigcup_{h_1^{(\delta_1)} \in \mathbf{h}_1} \{(h_1^{(\delta_1)})^\lambda\}$$

$$= \bigcup_{(\gamma_1^{(\delta_1)}, ..., \gamma_m^{(\delta_1)}) \in \mathbf{h}_1} \{([\gamma_1^{(\delta_1)}]^\lambda, ..., [\gamma_m^{(\delta_1)}]^\lambda]\}, \lambda > 0.$$
3 Correlation coefficients for EHFSs

In this section, we first briefly review the existing correlation coefficients proposed by Lu and Liang [21] for
EHFSs, and then, a new type of EHFS correlation coefficient is introduced following by a number of its prop-
erties. Eventually, this section ends with presenting the concept of weighted EHFS.

Let $X = \{x_1, x_2, ..., x_N\}$ be the reference set, $\mathcal{H}_1 = \{\langle x_i, h_1(x_i) \rangle | x_i \in X\}$

$$= \{\langle x_i, \bigcup_{\gamma(x_i) \in h_1(x_i)} \{\langle \gamma_1^1(x_i), ..., \gamma_m^l(x_i) \rangle \} | x_i \in X\}$$

$$\mathcal{H}_2 = \{\langle x_i, h_2(x_i) \rangle | x_i \in X\}$$

$$= \{\langle x_i, \bigcup_{\gamma(x_i) \in h_2(x_i)} \{\langle \gamma_1^1(x_i), ..., \gamma_m^l(x_i) \rangle \} | x_i \in X\}$$

be two EHFSs on $X$. When the EHFEs $h_1(x_i)$
and $h_2(x_i)$ are arranged in decreasing order of their elements and have the same length $T = \max\{l_{h_1}, l_{h_2}\}$, Lu
and Liang [21] introduced the below coefficient:

$$C_{EHFS_1}(\mathcal{H}_1, \mathcal{H}_2)$$

$$= \frac{1}{N} \sum_{i=1}^{N} S(h_1(x_i) \otimes h_2(x_i))$$

$$= \frac{1}{N} \sum_{i=1}^{N} S\left(\bigcup_{h_1^{(\delta_1)}(x_i) \in h_1(x_i), h_2^{(\delta_2)}(x_i) \in h_2(x_i)} \{h_1^{(\delta_1)}(x_i)h_2^{(\delta_2)}(x_i)\}\right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} S\left(\bigcup_{(\gamma_1^{1(\delta_1)}(x_i), ..., \gamma_m^{1(\delta_1)}(x_i)) \in h_1(x_i), (\gamma_1^{2(\delta_2)}(x_i), ..., \gamma_m^{2(\delta_2)}(x_i)) \in h_2(x_i)} \{\langle \gamma_1^{1(\delta_1)}(x_i), \gamma_1^{2(\delta_2)}(x_i), ..., \gamma_m^{1(\delta_1)}(x_i), \gamma_m^{2(\delta_2)}(x_i) \rangle \}\right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} S\left(\sum_{\delta_1, \delta_2} \{\langle(\gamma_1^{1(\delta_1)}(x_i), \gamma_1^{2(\delta_2)}(x_i), ..., \gamma_m^{1(\delta_1)}(x_i), \gamma_m^{2(\delta_2)}(x_i)\rangle\}\right)$$

(7)

where the function $S$ returns the summation of all its arguments.

Furthermore, in the case where the weights of decision makers are available, $\omega_k \geq 0 (k = 1, ..., m)$ with
$\sum_{k=1}^{m} \omega_k = 1$, and vector $W = (w_1, ..., w_N)$ indicates the weight vector of the attributes $x_i$ ($i = 1, ..., N$),
with properties $\sum_{i=1}^{N} w_i = 1$ and $w_i \geq 0 (i = 1, ..., N)$, Lu and Liang [21] extended their previous correlation
definition by using the expression:

\[ C_{WEHFS1}(\omega_1, \omega_2) = \sum_{i=1}^{N} w_i s(\omega_1 h_1(x_i) \otimes \omega_2 h_2(x_i)) \]

\[ = \sum_{i=1}^{N} w_i s( \bigcup_{h_1^{(1)}(x_i) \in h_1(x_i), h_2^{(2)}(x_i) \in h_2(x_i)} \{\omega_1 h_1^{(1)}(x_i) \otimes \omega_2 h_2^{(2)}(x_i)\}) \]

\[ = \sum_{i=1}^{N} w_i s( \sum_{\delta_1, \delta_2=1}^{T} \{ (\omega_1 \gamma_1^{(1)}(x_i), \omega_2 \gamma_2^{(2)}(x_i), \ldots, \omega_m \gamma_m^{(1)}(x_i), \omega_m \gamma_m^{(2)}(x_i)) \}, \]

(8)

where \( \omega_1 = \{ \langle x_i, \omega h_1(x_i) \rangle \mid x_i \in X \} \)

\( = \{ \langle x_i, \bigcup_{\omega_1 \gamma_1^{(1)}(x_i), \ldots, \omega_m \gamma_m^{(1)}(x_i)} \rangle \in h_1(x_i) \mid x_i \in X \} \) and \( \omega_2 = \{ \langle x_i, \omega h_2(x_i) \rangle \mid x_i \in X \} \)

Subsequently, Lu and Liang [21] presented two correlation coefficients between weighted EHFSs (WEHFSs) \( \omega_1 \) and \( \omega_2 \)

\[ \rho_{WEHFS1}(\omega_1, \omega_2) = \frac{C_{WEHFS1}(\omega_1, \omega_2)}{C_{WEHFS1}(\omega_1, \omega_1) C_{WEHFS1}(\omega_2, \omega_2)}]^{\frac{1}{2}}, \]

(9)

\[ \rho_{WEHFS2}(\omega_1, \omega_2) = \frac{C_{WEHFS1}(\omega_1, \omega_2)}{\max\{C_{WEHFS1}(\omega_1, \omega_1), C_{WEHFS1}(\omega_2, \omega_2)\}}. \]

(10)

Lu and Liang [21] proved the following result:

**Theorem 3.1.** [21] The correlation coefficients between two WEHFSs \( \omega_1 \) and \( \omega_2 \) \( \rho_{WEHFS_i} \) \((i = 1, 2)\), which are given by (9) and (10), satisfy the following properties

\[ 0 \leq \rho_{WEHFS_i}(\omega_1, \omega_2) \leq 1; \]

(11)

\[ \rho_{WEHFS_i}(\omega_1, \omega_2) = \rho_{WEHFS_i}(\omega_2, \omega_1); \]

(12)

\[ \rho_{WEHFS_i}(\omega_1, \omega_2) = 1, \text{ if } \omega_1 = \omega_2. \]

(13)

The following observations are noticed regarding Lu and Liang [21]’s correlation coefficients given by (9) and (10):
1. Each WEHFE $\omega h_j(x_i)$ (for $j = 1, 2$) is arranged in decreasing order;

2. The WEHFEs of WEHFSs $\omega H_1$ and $\omega H_2$ have the same length; and

3. The correlation coefficients return values in the interval $[0, 1]$, and ignore the negative correlation.

The above three observations motivates the developing of new correlation coefficients for WEHFSs.

3.1 Correlation coefficient between EHFSs

Prior to presenting the concept of correlation coefficient between EHFSs, we first recall some required definitions.

We begin with the definition of the mean value of an EHFE $h_1(x_i) = \{h_1^{(\delta_1)}(x_i) : = (\gamma_1^{1,(\delta_1)}(x_i), ..., \gamma_m^{1,(\delta_1)}(x_i)) \mid \delta_1 = 1, ..., l_{h_1} \}$ with respect to each $x_i \in X$

$$\bar{h}_1(x_i) = \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \bigcup_{h_1^{(\delta_1)}(x_i) \in h_1(x_i)} \{h_1^{(\delta_1)}(x_i)\}$$

$$= \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \bigcup_{(\gamma_1^{1,(\delta_1)}(x_i), ..., \gamma_m^{1,(\delta_1)}(x_i)) \in h_1(x_i) \}} \{(\gamma_1^{1,(\delta_1)}(x_i), ..., \gamma_m^{1,(\delta_1)}(x_i))\}$$

$$= \bigcup_{(\gamma_1^{1,(\delta_1)}(x_i), ..., \gamma_m^{1,(\delta_1)}(x_i)) \in h_1(x_i) \}} \{(1/l_{h_1}) \sum_{\delta_1=1}^{l_{h_1}} \gamma_1^{1,(\delta_1)}(x_i), ..., (1/l_{h_1}) \sum_{\delta_1=1}^{l_{h_1}} \gamma_m^{1,(\delta_1)}(x_i)\}$$

$$= (\bar{\gamma}_1(x_i), ..., \bar{\gamma}_m(x_i)). \quad (14)$$

The mean value of an EHFE is the m-tuple vector of mean values of its elements.
The variance value of an EHFE $\mathcal{H}_1 = \{x_i, h_1(x_i)\} \mid x_i \in X = \{(x_i, \cup_{\gamma_1(x_i), \ldots, \gamma_m(x_i) \in h_1(x_i)} \{(\gamma_1(x_i), \ldots, \gamma_m(x_i))\}) \mid x_i \in X\}$ is defined as:

$$E(\mathcal{H}_1) = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \bigcup \{h_1^{d_1}(x_i)\} \right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \left( \cup \{h_1^{d_1}(x_i)\} \right) - E(\mathcal{H}_1) \right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \left( \cup \{h_1^{d_1}(x_i)\} \right) - E(\mathcal{H}_1) \right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \left( \cup \{h_1^{d_1}(x_i)\} \right) - E(\mathcal{H}_1) \right]$$

$$= \left( E_{\gamma_1}, \ldots, E_{\gamma_{m_1}} \right).$$

The variance value of an EHFE is

$$V(\mathcal{H}_1) = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \left( \cup \{h_1^{d_1}(x_i)\} \right) - E(\mathcal{H}_1) \right]$$

The variance value of an EHFE is the m-tuple vector of variance values of its elements.

We introduce below, the new concept of correlation between two EHFSs $\mathcal{H}_1 = \{x_i, h_1(x_i)\} \mid x_i \in X = \{(x_i, \cup_{\gamma_1(x_i), \ldots, \gamma_m(x_i) \in h_1(x_i)} \{(\gamma_1(x_i), \ldots, \gamma_m(x_i))\}) \mid x_i \in X\}$ and $\mathcal{H}_2 = \{x_i, h_2(x_i)\} \mid x_i \in X$. 

10
\[
\{ \langle x_i, \mathbf{h}_1(x_i) \rangle \mid x_i \in X \} = \{ \langle x_i, \mathbf{h}_1(x_i) \rangle \in \mathbf{h}_2(x_i), \{\gamma_1^2(x_i), \ldots, \gamma_m^2(x_i)\} \mid x_i \in X \};
\]

\[
C(\mathcal{H}_1, \mathcal{H}_2) = \frac{1}{N} \sum_{i=1}^{N} [\mathbf{h}_1(x_i) - E(\mathcal{H}_1)][\mathbf{h}_2(x_i) - E(\mathcal{H}_2)]
\]  

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \mathbf{h}_1^{(d_1)}(x_i) - E(\mathcal{H}_1) \right] \cup \{ \mathbf{h}_2^{(d_2)}(x_i) - E(\mathcal{H}_2) \} \times [\frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \mathbf{h}_2^{(d_2)}(x_i) - E(\mathcal{H}_2)]
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \sum_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i))} \{\frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \gamma_1^{(d_1)}(x_i), \ldots, \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \gamma_m^{(d_1)}(x_i)\} - E(\mathcal{H}_1) \right] \cup \{\frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \gamma_1^{(d_2)}(x_i), \ldots, \frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \gamma_m^{(d_2)}(x_i)\} - E(\mathcal{H}_2)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \sum_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i))} \{\frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \gamma_i^{(d_1)}(x_i), \ldots, \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \gamma_m^{(d_1)}(x_i)\} - E(\mathcal{H}_1) \right] \cup \{\frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \gamma_i^{(d_2)}(x_i), \ldots, \frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \gamma_m^{(d_2)}(x_i)\} - E(\mathcal{H}_2)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \sum_{i=1}^{m} \gamma_i^2(x_i) - E_\gamma^2 \right] \cup \{\frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \gamma_i^{(d_1)}(x_i), \ldots, \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \gamma_m^{(d_1)}(x_i)\} - E(\mathcal{H}_1) \cup \{\frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \gamma_i^{(d_2)}(x_i), \ldots, \frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \gamma_m^{(d_2)}(x_i)\} - E(\mathcal{H}_2)
\]

(17)

The correlation value of an EHFE is the m-tuple vector of correlation values of its elements.

It is observed here that there is a relationship between variance as per expression (16) and the new correlation in the form of

\[
V(\mathcal{H}_1) = C(\mathcal{H}_1, \mathcal{H}_1),
\]  

for any EHFS \( \mathcal{H}_1 = \{ \langle x_i, \mathbf{h}_1(x_i) \rangle \mid x_i \in X \} = \{ \langle x_i, \cup_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i))} \rangle \mid \{\gamma_1^2(x_i), \ldots, \gamma_m^2(x_i)\}, x_i \in X \} \). Property (18) is consistent with the property of correlation and variance in the case of all sets being free of hesitancy, i.e. when they are classical sets.

The following lemma will be an entry point for the next discussions.
Lemma 3.2. Let
\[
H_c^r = \{ \langle x_i, h_c^r(x_i) \rangle \mid x_i \in X \}
\]
\[
= \{ x_i, \bigcup_{(\gamma_1(x_i), \ldots, \gamma_m(x_i)) \in h_1(x_i)} (1 - \gamma_1^1(x_i), \ldots, 1 - \gamma_m^1(x_i)) \} \mid x_i \in X \}
\]
be the complement of the EHFS
\[
H_1 = \{ \langle x_i, h_1(x_i) \rangle \mid x_i \in X \}
\]
\[
= \{ x_i, \bigcup_{(\gamma_1^1(x_i), \ldots, \gamma_m^1(x_i)) \in h_1(x_i)} (\gamma_1^1(x_i), \ldots, \gamma_m^1(x_i)) \} \mid x_i \in X \}.
\]
Then, it is
\[
E(H^r_c) = I_m - E(H_1) := (1 - E_{\gamma_1^1}, \ldots, 1 - E_{\gamma_m^1});
\] (19)
\[
C(H_1, H^r_c) = -C(H_1, H_1); \] (20)
\[
C(H^r_c, H^r_c) = C(H_1, H_1). \] (21)

Proof. Applying expression (15) to the EHFS \( H_1 = \{ \langle x_i, h_1(x_i) \rangle \mid x_i \in X \} \)
\[
\mathcal{H}_i = \{ x_i, \bigcup_{(\gamma_1^{(i)}(x_i), \ldots, \gamma_m^{(i)}(x_i)) \in \mathbf{h}_i(x_i)} \{ (\gamma_1^{(i)}(x_i), \ldots, \gamma_m^{(i)}(x_i)) \} \mid x_i \in X \}, \\
E(\mathcal{H}_i) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{h}_i(x_i) = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \mathbf{h}_1^{(\delta_1)}(x_i) \right) \bigcup \{ \mathbf{h}_i^{(\delta_1)}(x_i) \}
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} [1 - \gamma_1^{(i)}(x_i)], \ldots, \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} [1 - \gamma_m^{(i)}(x_i)] \right) \right) \bigcup \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \gamma_1^{(i)}(x_i), \ldots, \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \gamma_m^{(i)}(x_i) \right) \bigcup \{ (1, \ldots, 1) - \frac{1}{N} \sum_{i=1}^{N} \gamma_1^{(i)}(x_i), \ldots, \frac{1}{N} \sum_{i=1}^{N} \gamma_m^{(i)}(x_i) \}
\]

\[
= (1, \ldots, 1) - \frac{1}{N} \sum_{i=1}^{N} \gamma_1^{(i)}(x_i), \ldots, \frac{1}{N} \sum_{i=1}^{N} \gamma_m^{(i)}(x_i) \)
\]

\[
C(H_1, H_1) = \frac{1}{N} \sum_{i=1}^{N} [1_1(x_i) - E(H_1)][1_m(x_i) - E(H_1)]
\]

\[
= -\frac{1}{N} \sum_{i=1}^{N} [1_1(x_i) - E(H_1)][1_m(x_i) - E(H_1)] = -C(H_1, H_1), \quad (23)
\]

and subsequently,

\[
C(H_1^\dagger, H_1) = C(H_1, H_1). \quad \square \quad (24)
\]

In the below, new correlation coefficients between EHFSs are developed in response to the criticisms of Lu and Liang [21]'s correlation coefficients (9) and (10), while their properties are proved theoretically.
Definition 3.3. Let \( X = \{x_1, \ldots, x_N\} \) be the reference set, \( \mathcal{H}_1 = \{x_i, h_1(x_i) \mid x_i \in X\} \)
\( = \{x_i, \bigcup_{\gamma_1(x_i), \ldots, \gamma_m(x_i)} \{\{\gamma_1(x_i), \ldots, \gamma_m(x_i)\} \mid x_i \in X\} \)
and \( \mathcal{H}_2 = \{x_i, h_2(x_i) \mid x_i \in X\} \)
be two EHFSs. Therefore,

\[
\varrho_1(\mathcal{H}_1, \mathcal{H}_2) = \frac{C(\mathcal{H}_1, \mathcal{H}_2)}{(C(\mathcal{H}_1, \mathcal{H}_1), C(\mathcal{H}_2, \mathcal{H}_2))^{\frac{1}{2}}},
\]
\[
\varrho_2(\mathcal{H}_1, \mathcal{H}_2) = \frac{C(\mathcal{H}_1, \mathcal{H}_2)}{\max\{C(\mathcal{H}_1, \mathcal{H}_1), C(\mathcal{H}_2, \mathcal{H}_2)\}},
\]
\[
\varrho_{3, \lambda}(\mathcal{H}_1, \mathcal{H}_2) = \frac{C(\mathcal{H}_1, \mathcal{H}_2)}{\lambda(C(\mathcal{H}_1, \mathcal{H}_1), C(\mathcal{H}_2, \mathcal{H}_2))^{\frac{1}{2}} + (1 - \lambda) \max\{C(\mathcal{H}_1, \mathcal{H}_1), C(\mathcal{H}_2, \mathcal{H}_2)\}},
\]
\( 0 \leq \lambda \leq 1, \)

define correlation coefficients for EHFSs.

Corollary 3.4. In the case where \( \lambda = 1, 0 \), the correlation coefficient \( \varrho_{3, \lambda} \) becomes \( \varrho_1 \) and \( \varrho_2 \), respectively.

Proof. The proof is immediate from the correlation coefficient definitions (25), (26) and (27).

Theorem 3.5. The correlation coefficients \( \varrho_1, \varrho_2 \) and \( \varrho_{3, \lambda} \) satisfy

\[
-1 \leq \rho_i(\mathcal{H}_1, \mathcal{H}_2) \leq 1; \quad \text{(28)}
\]
\[
\rho_i(\mathcal{H}_1, \mathcal{H}_2) = \rho_i(\mathcal{H}_2, \mathcal{H}_1); \quad \text{(29)}
\]
\[
\rho_i(\mathcal{H}_1, \mathcal{H}_1) = 1; \quad \text{(30)}
\]
\[
\rho_i(\mathcal{H}_1, \mathcal{H}_1^c) = -1. \quad \text{(31)}
\]

Proof. To prove (28), three distinct cases based on equations (25), (26) and (27) are considered.
Case 1. From (25) and (17), it is

\[ |C(H_1, H_2)| = \frac{1}{N} \sum_{i=1}^{N} |\mathbf{h}_1(x_i) - E(H_1)| |\mathbf{h}_2(x_i) - E(H_2)| \]

\[ \leq \frac{1}{N} \sum_{i=1}^{N} \left| \frac{1}{l_{h_1}} \sum_{h_1} \mathbf{h}_1(x_i) - E(H_1) \right| \times |\mathbf{h}_2(x_i) - E(H_2)| \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{1}{l_{h_1}} \sum_{h_1} \mathbf{h}_1(x_i) \right| \times \left| \frac{1}{l_{h_2}} \sum_{h_2} \mathbf{h}_2(x_i) \right| \]

\[ \leq \frac{1}{N} \sum_{i=1}^{N} \left| \frac{1}{l_{h_1}} \sum_{h_1} \mathbf{h}_1(x_i) \right| \times \left| \frac{1}{l_{h_2}} \sum_{h_2} \mathbf{h}_2(x_i) \right| \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{1}{l_{h_1}} \sum_{h_1} \mathbf{h}_1(x_i) \right| \times \left| \frac{1}{l_{h_2}} \sum_{h_2} \mathbf{h}_2(x_i) \right| \]

Applying Cauchy-Schwarz inequality to the latter result, one concludes the following inequality

\[ |C(H_1, H_2)| \leq \frac{1}{N} \sum_{i=1}^{N} \left| \frac{1}{l_{h_1}} \sum_{h_1} \gamma_{1}^{1}(x_i) \right| \times \left| \frac{1}{l_{h_2}} \sum_{h_2} \gamma_{2}^{2}(x_i) \right| \]

\[ = (C(H_1, H_1))^{\frac{1}{2}} \times (C(H_2, H_2))^{\frac{1}{2}}. \]
Case 2. From (26), (17) and Case 1, it is

\[ |C(H_1, H_2)| \]
\[ = \left| \frac{1}{N} \sum_{i=1}^{N} [\mathbf{h}_1(x_i) - E(H_1)][\mathbf{h}_2(x_i) - E(H_2)] \right| \]
\[ \leq \frac{1}{N} \sum_{i=1}^{N} ||\mathbf{h}_1(x_i) - E(H_1)|| \times ||\mathbf{h}_2(x_i) - E(H_2)|| \]
\[ = \frac{1}{N} \sum_{i=1}^{N} \left( \left| \frac{1}{l_{h_1}} \sum_{d_1=1}^{l_{h_1}} \mathbf{h}_1^{(d_1)}(x_i) - E(H_1) \right| \right) \times \left( \left| \frac{1}{l_{h_2}} \sum_{d_2=1}^{l_{h_2}} \mathbf{h}_2^{(d_2)}(x_i) - E(H_2) \right| \right) \]
\[ = \frac{1}{N} \sum_{i=1}^{N} \left( \left| \sum_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i)) \in \mathbf{h}_1(x_i)} \right| \right) \times \left( \left| \sum_{(\gamma_1^{(d_2)}(x_i), \ldots, \gamma_m^{(d_2)}(x_i)) \in \mathbf{h}_2(x_i)} \right| \right) \]
\[ = \sum_{i=1}^{N} \sum_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i)) \in \mathbf{h}_1(x_i)} \sum_{(\gamma_1^{(d_2)}(x_i), \ldots, \gamma_m^{(d_2)}(x_i)) \in \mathbf{h}_2(x_i)} \]
\[ = \left( \sum_{i=1}^{N} \sum_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i)) \in \mathbf{h}_1(x_i)} \right) \times \left( \sum_{i=1}^{N} \sum_{(\gamma_1^{(d_2)}(x_i), \ldots, \gamma_m^{(d_2)}(x_i)) \in \mathbf{h}_2(x_i)} \right) \]
\[ = (C(H_1, H_1))^\frac{1}{2} \times (C(H_2, H_2))^\frac{1}{2} \]
\[ (36) \]

Applying the Cauchy-Schwarz inequality \(\sum_{k=1}^{N} a_k b_k \leq (\sum_{k=1}^{N} a_k^2)^\frac{1}{2}(\sum_{k=1}^{N} b_k^2)^\frac{1}{2}\) to the latter result, it can be seen that

\[ |C(H_1, H_2)| \leq \left( \frac{1}{N} \sum_{i=1}^{N} \left( \left| \sum_{(\gamma_1^{(d_1)}(x_i), \ldots, \gamma_m^{(d_1)}(x_i)) \in \mathbf{h}_1(x_i)} \right| \right) \right) \times \left( \left| \sum_{(\gamma_1^{(d_2)}(x_i), \ldots, \gamma_m^{(d_2)}(x_i)) \in \mathbf{h}_2(x_i)} \right| \right) \]
\[ = (C(H_1, H_1))^\frac{1}{2} \times (C(H_2, H_2))^\frac{1}{2} \]
\[ (37) \]

Moreover,

\[ |C(H_1, H_1) \times C(H_2, H_2)|^\frac{1}{2} \leq \left| \max\{ (C(H_1, H_1))^2, (C(H_2, H_2))^2 \} \right| \frac{1}{2} \]
\[ = \max\{ (C(H_1, H_1))^2, (C(H_2, H_2))^2 \} \]
\[ (38) \]

This implies that

\[ |\rho_2(H_1, H_2)| = \frac{|C(H_1, H_2)|}{\max\{ (C(H_1, H_1)), (C(H_2, H_2)) \}} \leq 1, \]

16
and therefore,

\[-1 \leq \rho_2(\mathcal{H}_1, \mathcal{H}_2) \leq 1.\]

**Case 3.** From (27), (17) and Cases 1 and 2, it is

\[|C(\mathcal{H}_1, \mathcal{H}_2)| \leq |C(\mathcal{H}_1, \mathcal{H}_1) \times C(\mathcal{H}_2, \mathcal{H}_2)|^{\frac{1}{2}} \leq \max\{C(\mathcal{H}_1, \mathcal{H}_1), C(\mathcal{H}_2, \mathcal{H}_2)\},\]  

(39)

and therefore,

\[|C(\mathcal{H}_1, \mathcal{H}_2)| \leq \lambda|C(\mathcal{H}_1, \mathcal{H}_1) \times C(\mathcal{H}_2, \mathcal{H}_2)|^{\frac{1}{2}}\]

\[+(1 - \lambda) \max\{C(\mathcal{H}_1, \mathcal{H}_1), C(\mathcal{H}_2, \mathcal{H}_2)\}, \quad (0 \leq \lambda \leq 1).\]  

(40)

The latter relation results in

\[-1 \leq \rho_{3,\lambda}(\mathcal{H}_1, \mathcal{H}_2) \leq 1.\]

The proofs of (29), (30) and (31) derive from Lemma 3.2. □

### 3.2 Weighted correlation coefficient between WEHFSs

This section extends the proposed correlation coefficients from EHFSs to WEHFSs by taking into account the weight of EHFSs.

Let us consider two WEHFSs \(\omega\mathcal{H}_1 = \{(x_i, \omega h_1(x_i)) \mid x_i \in X\} = \\
\{(x_i, \bigcup_{\omega \gamma_1(x_i)} \ldots \omega \gamma_m(x_i)) \mid x_i \in X\}\) and \(\omega\mathcal{H}_2 = \{(x_i, \omega h_2(x_i)) \mid x_i \in X\} = \\
\{(x_i, \bigcup_{\omega \gamma_1(x_i)} \ldots \omega \gamma_m(x_i)) \mid x_i \in X\}\) with weight vector \(w = (w_1, \ldots, w_N)\), where \(0 \leq w_i \leq 1\) and \(\sum_{i=1}^{N} w_i = 1\). The correlation coefficient is extended into the following weighted
correlation:

\[
C_W(\omega \mathcal{H}_1, \omega \mathcal{H}_2) = \sum_{i=1}^{N} w_i [\omega \tilde{h}_1(x_i) - E(\omega \mathcal{H}_1)][\omega \tilde{h}_2(x_i) - E(\omega \mathcal{H}_2)]
\]

\[= \sum_{i=1}^{N} w_i \]

\[
\frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \bigcup \{\omega h_1^{(\delta_1)}(x_i)\} - E(\omega \mathcal{H}_1) \bigcup \frac{1}{l_{h_2}} \sum_{\delta_2=1}^{l_{h_2}} \{\omega h_2^{(\delta_2)}(x_i)\} - E(\omega \mathcal{H}_2)
\]

\[= \sum_{i=1}^{N} w_i \]

\[
\bigcup (\omega \gamma_1^{(\delta_1)}(x_i), \ldots, \omega \gamma_m^{(\delta_1)}(x_i)) \subseteq \omega \mathcal{H}_1(x_i)
\]

\[
\bigcup (\omega \gamma_1^{(\delta_2)}(x_i), \ldots, \omega \gamma_m^{(\delta_2)}(x_i)) \subseteq \omega \mathcal{H}_2(x_i)
\]

\[= \sum_{i=1}^{N} w_i \]

\[
[(\omega \gamma_1^{m_1}(x_i), \ldots, \omega \gamma_m^{m_1}(x_i)) - (E_{\omega \gamma_1^{m_1}}, \ldots, E_{\omega \gamma_m^{m_1}})[(\omega \gamma_1^{m_2}(x_i), \ldots, \omega \gamma_m^{m_2}(x_i)) - (E_{\omega \gamma_1^{m_2}}, \ldots, E_{\omega \gamma_m^{m_2}})]
\]

\[= \sum_{i=1}^{N} w_i \]

\[
[(\omega \gamma_1^{m_1}(x_i) - E_{\omega \gamma_1^{m_1}}, \ldots, \omega \gamma_m^{m_1}(x_i) - E_{\omega \gamma_m^{m_1}})[(\omega \gamma_1^{m_2}(x_i) - E_{\omega \gamma_1^{m_2}}, \ldots, \omega \gamma_m^{m_2}(x_i) - E_{\omega \gamma_m^{m_2}})]
\]

\[= \sum_{i=1}^{N} w_i \]

\[
[ (\omega \gamma_1^{m_1}(x_i) - E_{\omega \gamma_1^{m_1}})(\omega \gamma_1^{m_2}(x_i) - E_{\omega \gamma_1^{m_2}}, \ldots, (\omega \gamma_m^{m_1}(x_i) - E_{\omega \gamma_m^{m_1}})(\omega \gamma_m^{m_2}(x_i) - E_{\omega \gamma_m^{m_2}})]
\]

\[= \sum_{i=1}^{N} w_i \]

\[\sum_{i=1}^{N} w_i[\omega \gamma_1^{m_1}(x_i) - E_{\omega \gamma_1^{m_1}})(\omega \gamma_1^{m_2}(x_i) - E_{\omega \gamma_1^{m_2}}, \ldots, (\omega \gamma_m^{m_1}(x_i) - E_{\omega \gamma_m^{m_1}})(\omega \gamma_m^{m_2}(x_i) - E_{\omega \gamma_m^{m_2}})]
\]

\[= (\sum_{i=1}^{N} w_i) \]

\[
\sum_{i=1}^{N} w_i[\omega \gamma_1^{m_1}(x_i) - E_{\omega \gamma_1^{m_1}})(\omega \gamma_1^{m_2}(x_i) - E_{\omega \gamma_1^{m_2}}, \ldots, (\omega \gamma_m^{m_1}(x_i) - E_{\omega \gamma_m^{m_1}})(\omega \gamma_m^{m_2}(x_i) - E_{\omega \gamma_m^{m_2}})]
\]

\[= (\sum_{i=1}^{N} w_i) \]

\[
\sum_{i=1}^{N} w_i[\omega \gamma_1^{m_1}(x_i) - E_{\omega \gamma_1^{m_1}})(\omega \gamma_1^{m_2}(x_i) - E_{\omega \gamma_1^{m_2}}, \ldots, (\omega \gamma_m^{m_1}(x_i) - E_{\omega \gamma_m^{m_1}})(\omega \gamma_m^{m_2}(x_i) - E_{\omega \gamma_m^{m_2}})]
\]

\[= (\sum_{i=1}^{N} w_i) \]
Moreover, we define

$$E_W(\omega H_1) = \sum_{i=1}^{N} w_i \mathbb{I}_1(x_i) = \sum_{i=1}^{N} w_i \left[ \frac{1}{l_{h_1}} \sum_{\delta_i=1}^{l_{h_1}} \mathbb{I}_1^{-\omega h_1}(x_i) \right]$$

$$= \sum_{i=1}^{N} w_i \left[ \mathbb{I}_1^{-\omega h_1}(x_i) \right] = \frac{1}{l_{h_1}} \sum_{\delta_i=1}^{l_{h_1}} \omega h_1^{-1}(x_i)$$

$$= \sum_{i=1}^{N} w_i \left[ \omega h_1^{-1}(x_i) \right] = \left( \sum_{i=1}^{N} w_i \right)^{-1}(x_i)$$

$$= (E_{W\omega h_1}^1, ..., E_{W\omega h_1}^N).$$

Taking the definition of $C_W(\omega H_1, \omega H_2)$ and $E_W(\omega H_1)$ into consideration, the corresponding weighted correlation coefficients between WEHFSs $\omega H_1$ and $\omega H_2$ are defined as

$$\rho_{W1}(\omega H_1, \omega H_2) = \frac{C_W(\omega H_1, \omega H_2)}{(C_W(\omega H_1, \omega H_1) \times C_W(\omega H_2, \omega H_2))^{\frac{1}{2}}}$$

$$\rho_{W2}(\omega H_1, \omega H_2) = \frac{C_W(\omega H_1, \omega H_2)}{\max\{C_W(\omega H_1, \omega H_1), C_W(\omega H_2, \omega H_2)\}}$$

$$\rho_{W3}(\omega H_1, \omega H_2) = \frac{C_W(\omega H_1, \omega H_2)}{\lambda(C_W(\omega H_1, \omega H_1) \times C_W(\omega H_2, \omega H_2))^{\frac{1}{2}} + (1 - \lambda) \max\{C_W(\omega H_1, \omega H_1), C_W(\omega H_2, \omega H_2)\}}$$

for any $0 \leq \lambda \leq 1$.

Now, we are in a position to state our main research results.

**Lemma 3.6.** Let $\omega H_1^c = \{ (x_i, \omega h_1^c(x_i)) \mid x_i \in X \} = \{ (x_i, \cup_{\omega h_1^c(x_i)}(x_i)) \mid x_i \in X \}$ be the complement of the WEHFS $\omega H_1 = \{ (x_i, \omega h_1(x_i)) \mid x_i \in X \}$

$$= \{ (x_i, \cup_{\omega h_1^c(x_i)}(x_i)) \mid x_i \in X \}.$$ Then, it is

$$E_W(\omega H_1^c) = I_m - E_W(\omega H_1) := (1 - E_{W\omega h_1^c}, ..., 1 - E_{W\omega h_1^c})$$

$$C_W(\omega H_1^c, \omega H_1^c) = -C_W(\omega H_1, \omega H_1);$$

$$C_W(\omega H_1^c, \omega H_1^c) = C_W(\omega H_1, \omega H_1).$$
Proof. Here, only the first equation (46) is proved as the others are proved similarly. By applying expression (42) to the WEHFS \( \omega \mathcal{H}_1 = \{ (x_i, \omega \mathbf{h}_1(x_i)) \mid x_i \in X \} \)

\[ \{ (x_i, \bigcup_{\omega \gamma_1^{1}(x_i), \ldots, \omega \gamma_m^{1}(x_i) \in \omega \mathbf{h}_1(x_i)} \{ \omega \gamma_1^{1}(x_i), \ldots, \omega \gamma_m^{1}(x_i) \}) \mid x_i \in X \} \]

it can be deduced that

\[
E_W(\omega \mathcal{H}_1^c) = \sum_{i=1}^{N} w_i^c \mathbf{h}_i(x_i) = \sum_{i=1}^{N} w_i \left[ \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \left\{ \omega \mathbf{h}_1^{(\delta_1)}(x_i) \right\} \right]
\]

\[
= \sum_{i=1}^{N} w_i \left[ \bigcup_{\omega \gamma_1^{1}(x_i), \ldots, \omega \gamma_m^{1}(x_i) \in \omega \mathbf{h}_1(x_i)} \left\{ \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} [1 - \omega \gamma_1^{1}(\delta_1)(x_i)], \ldots, \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} [1 - \omega \gamma_m^{1}(\delta_1)(x_i)] \right) \right\} \right]
\]

\[
= \sum_{i=1}^{N} w_i \left[ \bigcup_{\omega \gamma_1^{1}(x_i), \ldots, \omega \gamma_m^{1}(x_i) \in \omega \mathbf{h}_1(x_i)} \left\{ \left(1, \ldots, 1\right) - \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \omega \gamma_1^{1}(\delta_1)(x_i), \ldots, \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \omega \gamma_m^{1}(\delta_1)(x_i) \right) \right\} \right]
\]

\[
= \left(1, \ldots, 1\right) - \sum_{i=1}^{N} w_i \left[ \bigcup_{\omega \gamma_1^{1}(x_i), \ldots, \omega \gamma_m^{1}(x_i) \in \omega \mathbf{h}_1(x_i)} \left\{ \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \omega \gamma_1^{1}(\delta_1)(x_i), \ldots, \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \omega \gamma_m^{1}(\delta_1)(x_i) \right) \right\} \right]
\]

\[
= \left(1, \ldots, 1\right) - \sum_{i=1}^{N} w_i \left[ \omega \gamma_1 \left( x_i, \ldots, \omega \gamma_m \left( x_i \right) \right) \right] = \left(1, \ldots, 1\right) - \left( \sum_{i=1}^{N} \omega \gamma_1 \left( x_i, \ldots, \omega \gamma_m \left( x_i \right) \right) \right)
\]

\[
= 1_m - E_W(\omega \mathcal{H}_1) := (1 - E_W \gamma_1^1, \ldots, 1 - E_W \gamma_m^1).
\] (49)

Moreover,

\[
V(\mathcal{H}_1^c) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{h}_i(x_i) - E(\mathcal{H}_1^c) = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} \mathbf{h}_1^{(\delta_1)}(x_i) \right] - \left( \sum_{i=1}^{N} \mathbf{h}_1^{(\delta_1)}(x_i) \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \bigcup_{\gamma_1^{1}(x_i), \ldots, \gamma_m^{1}(x_i) \in \mathbf{h}_1(x_i)} \left\{ \left( \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} [1 - \gamma_1^{1}(\delta_1)(x_i)], \ldots, \frac{1}{l_{h_1}} \sum_{\delta_1=1}^{l_{h_1}} [1 - \gamma_m^{1}(\delta_1)(x_i)] \right) \right\} \right]
\]

\[
= \left[ 1_m - E(\mathcal{H}_1) \right]
\]

\[
= \left[ \frac{1}{N} \sum_{i=1}^{N} \gamma_1^1(x_i), \ldots, \frac{1}{N} \sum_{i=1}^{N} \gamma_m^1(x_i) \right] - \left( E \gamma_1^1, \ldots, E \gamma_m^1 \right)
\]

\[
= (V \gamma_1^1, \ldots, V \gamma_m^1) = V(\mathcal{H}_1).
\] (50)
As a result, it is
\[
[h_1(x_i) - E(H_1^c)] = [(1_m - h_1(x_i)) - (1_m - E(H_1))] = -[h_1(x_i) - E(H_1)],
\]
which yields
\[
C(H_1, H_1^c) = \frac{1}{N} \sum_{i=1}^{N} [h_1(x_i) - E(H_1)][h_1^c(x_i) - E(H_1^c)]
\]
\[
= -\frac{1}{N} \sum_{i=1}^{N} [h_1(x_i) - E(H_1)][h_1(x_i) - E(H_1)] = -C(H_1, H_1), \tag{51}
\]
and subsequently,
\[
C(H_1^c, H_1^c) = C(H_1, H_1). \tag{52}
\]

**Theorem 3.7.** The weighted correlation coefficients between two WEHFSs $\omega H_1$ and $\omega H_2$ $\rho_{W_i}$ ($i = 1, 2, (3, \lambda)$) satisfy the following properties

\[
-1 \leq \rho_{W_i}(\omega H_1, \omega H_2) \leq 1; \tag{53}
\]
\[
\rho_{W_i}(\omega H_1, \omega H_2) = \rho_i(\omega H_2, \omega H_1); \tag{54}
\]
\[
\rho_{W_i}(\omega H_1, \omega H_1) = 1; \tag{55}
\]
\[
\rho_{W_i}(\omega H_1, \omega H_1^c) = -1. \tag{56}
\]

**Proof.** Using Lemma 3.6, the proof is similar to the proof of Theorem 3.7, and it is omitted. \qed

Lu and Liang [21] also introduced the following kind of weighted correlation coefficients of WEHFSs without
\[
\rho_{\text{WEHFS}3}(\omega H_1, \omega H_2) = \frac{C_{\text{WEHFS}2}(\omega H_1, \omega H_2)}{\left( e_{\text{WEHFS}2}(\omega H_1) e_{\text{WEHFS}2}(\omega H_2^H) \right)^{\frac{1}{2}} + \left( e_{\text{WEHFS}2}(\omega H_2) e_{\text{WEHFS}2}(\omega H_1^H) \right)^{\frac{1}{2}}} \]  \hspace{1cm} (57)

\[
\rho_{\text{WEHFS}4}(\omega H_1, \omega H_2) = \frac{1}{2} \left[ \frac{C_{\text{WEHFS}3}(\omega H_1, \omega H_2)}{\left( e_{\text{WEHFS}2}(\omega H_1) e_{\text{WEHFS}2}(\omega H_2^H) \right)^{\frac{1}{2}}} + \frac{C_{\text{WEHFS}3}(\omega H_1, \omega H_2)}{\left( e_{\text{WEHFS}2}(\omega H_2) e_{\text{WEHFS}2}(\omega H_1^H) \right)^{\frac{1}{2}}} \right] \hspace{1cm} (58)
\]

\[
\rho_{\text{WEHFS}5}(\omega H_1, \omega H_2) = \frac{C_{\text{WEHFS}2}(\omega H_1, \omega H_2)}{\max\{e_{\text{WEHFS}2}(\omega H_1), e_{\text{WEHFS}2}(\omega H_2^H)\} + \max\{e_{\text{WEHFS}2}(\omega H_2), e_{\text{WEHFS}2}(\omega H_1^H)\}} \hspace{1cm} (59)
\]

\[
\rho_{\text{WEHFS}6}(\omega H_1, \omega H_2) = \frac{1}{2} \left[ \frac{C_{\text{WEHFS}3}(\omega H_1, \omega H_2)}{\max\{e_{\text{WEHFS}2}(\omega H_1), e_{\text{WEHFS}2}(\omega H_2^H)\}} + \frac{C_{\text{WEHFS}3}(\omega H_1, \omega H_2)}{\max\{e_{\text{WEHFS}2}(\omega H_2), e_{\text{WEHFS}2}(\omega H_1^H)\}} \right]. \hspace{1cm} (60)
\]
where

\[ e_{\text{WEHFS}_2}(\omega \mathcal{H}_1) = \sum_{i=1}^{N} w_i \mathbf{S}(\sigma \mathbf{h}_1(x_i) \otimes \sigma \mathbf{h}_1(x_i)) = \sum_{i=1}^{N} w_i \mathbf{S}(\bigcup_{\gamma_1(x_i) \in \mathbf{h}_1(x_i)} \{\omega \mathbf{h}_1^{(\delta_1)}(x_i) \circ \omega \mathbf{h}_1^{(\delta_2)}(x_i)\}) \]

\[ = \sum_{i=1}^{N} w_i \mathbf{S}(\bigcup_{\delta_1=1}^{T} \{[\omega_1 \gamma_1^{(\delta_1)}(x_i)]^2, \ldots, [\omega_m \gamma_m^{(\delta_1)}(x_i)]^2\}) \]

\[ e_{\text{WEHFS}_2}(\omega \mathcal{H}_2) = \sum_{i=1}^{N} w_i \mathbf{S}(\sigma \mathbf{h}_1(x_i) \otimes \sigma \mathbf{h}_1(x_i)) = \sum_{i=1}^{N} w_i \mathbf{S}(\bigcup_{\gamma_1(x_i) \in \mathbf{h}_1(x_i)} \{\omega \mathbf{h}_1^{(\delta_1)}(x_i) \circ \omega \mathbf{h}_1^{(\delta_2)}(x_i)\}) \]

\[ = \sum_{i=1}^{N} w_i \mathbf{S}(\bigcup_{\delta_1=1}^{T} \{[\omega_1 \gamma_1^{(\delta_1)}(x_i)]^2, \ldots, [\omega_m \gamma_m^{(\delta_1)}(x_i)]^2\}) \]

\[ = e_{\text{WEHFS}_2}(\omega \mathcal{H}_1) + e_{\text{WEHFS}_2}(\omega \mathcal{H}_2) \]

with \( \omega \mathbf{h}_2^{(\delta_2)}(x_i) \) selected from all \( \omega \mathbf{h}_1^{(\delta_2)}(x_i) \) with minimum distance from \( \omega \mathcal{H}_1 \) (see Definition 15 in [21]),

\[ \omega \mathcal{H}_1 = \{\langle x_i, \omega \mathbf{h}_1(x_i) \rangle \mid x_i \in X\} = \{\langle x_i, \bigcup_{\gamma_1(x_i)} \{[\omega_1 \gamma_1^{(\delta_1)}(x_i)]^2, \ldots, [\omega_m \gamma_m^{(\delta_1)}(x_i)]^2\}\rangle \mid x_i \in X\} \]

\[ \omega \mathcal{H}_2 = \{\langle x_i, \omega \mathbf{h}_2(x_i) \rangle \mid x_i \in X\} = \{\langle x_i, \bigcup_{\gamma_1(x_i)} \{[\omega_1 \gamma_1^{(\delta_1)}(x_i)]^2, \ldots, [\omega_m \gamma_m^{(\delta_1)}(x_i)]^2\}\rangle \mid x_i \in X\} \]

The above weighted correlation coefficients between two WEHFSs \( \omega \mathcal{H}_1 \) and \( \omega \mathcal{H}_2 \) \( \rho_{\text{WEHFS}_i} \) \( (i = 3, 4, 5, 6) \) (57)-(60) satisfy the following properties

\[ 0 \leq \rho_{\text{WEHFS}_i}(\omega \mathcal{H}_1, \omega \mathcal{H}_2) \leq 1; \]  \hspace{1cm} (61)

\[ \rho_{\text{WEHFS}_i}(\omega \mathcal{H}_1, \omega \mathcal{H}_2) = \rho_{\text{WEHFS}_i}(\omega \mathcal{H}_2, \omega \mathcal{H}_1); \]  \hspace{1cm} (62)

\[ \rho_{\text{WEHFS}_i}(\omega \mathcal{H}_1, \omega \mathcal{H}_2) = 1, \text{ if } \omega \mathcal{H}_1 = \omega \mathcal{H}_2. \]  \hspace{1cm} (63)

It is obvious that the proposed new weighted correlation coefficients \( \rho_{\mathcal{W}_i} \) \( (i = 1, 2, (3, \lambda)) \) have some advan-
tages, listed below as requirements that are used throughout the next discussions:

1. The weighted correlation coefficients $\rho_{Wi} (i = 1, 2, (3, \lambda))$ are computed without the need to re-arrange in decreasing order the WEHFSs $\prec H_1$ and $\prec H_2$ as the weighted correlation coefficients $\rho_{WEHFS_i} (i = 1, 2)$ require;

2. The weighted correlation coefficients $\rho_{Wi} (i = 1, 2, (3, \lambda))$ are computed without the need to extend the weighted corresponding WEHFEs of the two WEHFSs $\prec H_1$ and $\prec H_2$ to make them have equal length as required by $\rho_{WEHFS_i} (i = 1, 2)$; and

3. The proposed weighted correlation coefficients $\rho_{Wi} (i = 1, 2, (3, \lambda))$ range is the interval $[-1, 1]$, while the weighted correlation coefficients $\rho_{WEHFS_i} (i = 1, 2, 3, 4, 5, 6)$ range is the interval $[0, 1]$.

4 Decision making technique based on EHFSs

The aim of this section is to implement the proposed weighted correlation coefficients of EHFEs to decision making with extended hesitant fuzzy information for demonstrating the efficiency and effectiveness of the proposed correlation coefficients.

4.1 Application of correlation coefficients between EHFSs to the energy policy problem

Assume that $A = \{A_1, ..., A_n\}$ denotes a collection of $n$ alternatives that are evaluated by $m$ decision makers with corresponding weights $\omega_k \geq 0 (k = 1, ..., m)$ and $\sum_{k=1}^{m} \omega_k = 1$, using a collection of $N$ criteria $C = \{C_1, ..., C_N\}$ with weights $w_i \geq 0 (i = 1, ..., N)$ and $\sum_{i=1}^{N} w_i = 1$. 

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If $C^k_j(A_i)$ denotes the opinion of decision maker $D_k$ ($k = 1, ..., m$) provided for the alternative $A_i$ ($i = 1, ..., n$) with respect to the criterion $C_j$ ($j = 1, ..., N$), then the notation $\omega h_{ij} = \{\omega h_{ij}^\delta = (\omega_1 \gamma_{ij}, \delta, ..., \omega_m \gamma_{ij}, \delta) \mid \delta = 1, ..., l_{h_{ij}}\}$ is used to indicate the corresponding element of the weighted extended hesitant fuzzy decision matrix $\omega H = \{\omega h_{ij}\}_{i \times N}$.

In the same way as it was done in Zhu and Xu [43] and Lu and Liang [21], the ideal value $h_{ideal} = \{h_{ideal}^\delta = (1, ..., 1) \mid \delta = 1, ..., l_{h_{ideal}}\}$ is adopted for evaluating alternatives. Here, it should be noted that the ideal value is independent of the decision makers’ priority, and therefore, it is easily seen that

$$\omega h_{ideal}(x_i) := h_{ideal} = \{(1, ..., 1)\};$$

and, consequently

$$E_W(\omega H_{ideal}) := E(H_{ideal}) = \frac{1}{N} \sum_{i=1}^{N} h_{ideal}(x_i) = \frac{1}{N} \sum_{i=1}^{N} \{(1, ..., 1)\}$$

$$= \{(1, ..., 1)\}. \quad (65)$$

Furthermore, it is assumed that all the decision makers $D_k$ ($k = 1, ..., m$) have the same important, i.e., $\omega_1 = \omega_2 = ... = \omega_m$. This assumption makes $\omega h_{ideal}$ rather simple in the form of $\omega h_{ideal} = \{\omega h_{ideal}^\delta = \frac{1}{m}(1, ..., 1) \mid \delta = 1, ..., l_{h_{ideal}}\}$. For the sake of simplicity, we hereafter suppose that $\omega h_{ideal} = \{\omega h_{ideal}^\delta = (1, ..., 1) \mid \delta = 1, ..., l_{h_{ideal}}\}$.

The algorithm that describes the procedure to select the best alternative(s) in a decision making problem under extended hesitant fuzzy environment was presented by Zhu and Xu [43] and Lu and Liang [21].

Algorithm 1.

Step 1 Construct the weighted extended hesitant fuzzy decision matrix $\omega H = \{\omega h_{ij}\}_{i \times N}$.

Step 2 The weighted correlation coefficients between an alternative $A_i$ ($i = 1, 2, ..., n$) and the ideal alternative $\omega h_{ideal}$ are computed.
Step 3. Rank all the alternatives on the basis of their weighted correlation coefficient values, and select the closest one to the ideal alternative $h_{ideal}$ as the optimal alternative.

The below deals with a social-economic developing of societies that is based on the energy as an indispensable factor, investigated by Zhu and Xu [43] and Lu and Liang [21]. It is assumed that five alternatives (i.e., energy projects) $A_i$ ($i = 1, 2, 3, 4, 5$) are assessed according to four criteria: $C_1$: technological, $C_2$: environmental, $C_3$: socio-political, and $C_4$: economic. Moreover, five anonymous decision makers evaluate the alternatives with respect to the above criteria and provide extended hesitant fuzzy decision matrices $H = [h_{ij}]_{5 \times N}$ as shown in Table 1. As in Zhu and Xu [43] and Lu and Liang’s [21] scenario, it is considered the priority vector of the decision makers is $\omega = (0.3, 0.1, 0.3, 0.2, 0.1)$.

<table>
<thead>
<tr>
<th>Criteria $C_1$</th>
<th>Criteria $C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy project $A_1$</td>
<td>$h_{11} = {0.3, 0.4, 0.3, 0.4, 0.5}$ $h_{12} = {0.7, 0.8, 0.3, 0.8, 0.6}$</td>
</tr>
<tr>
<td>Energy project $A_2$</td>
<td>$h_{21} = {0.3, 0.4, 0.5, 0.2, 0.5}$ $h_{22} = {0.5, 0.6, 0.5, 0.6, 0.6}$</td>
</tr>
<tr>
<td>Energy project $A_3$</td>
<td>$h_{31} = {0.4, 0.5, 0.5, 0.5, 0.6}$ $h_{32} = {0.5, 0.6, 0.7, 0.6, 0.5}$</td>
</tr>
<tr>
<td>Energy project $A_4$</td>
<td>$h_{41} = {0.3, 0.2, 0.2, 0.3, 0.1}$ $h_{42} = {0.6, 0.5, 0.7, 0.5, 0.5}$</td>
</tr>
<tr>
<td>Energy project $A_5$</td>
<td>$h_{51} = {0.3, 0.4, 0.6, 0.2, 0.2}$ $h_{52} = {0.6, 0.8, 0.5, 0.4, 0.6}$</td>
</tr>
</tbody>
</table>

Table 1. The extended hesitant fuzzy decision matrix $H = [h_{ij}]_{5 \times 4}$.

Taking into account Zhu and Xu [43]’s and Lu and Liang [21]’s criteria weighting vector $w = (0.3, 0.2, 0.2, 0.3)$, the above-described decision making problem is solved through the following steps of Algorithm 1:
Step 1. From the weighted extended hesitant fuzzy decision matrix $\omega H = [\omega h_{ij}]_{5 \times 4}$, the mean of the WEHFSs are calculated and shown in Table 2.

<table>
<thead>
<tr>
<th>Criteria $C_1$ ($w_1 = 0.3$)</th>
<th>Criteria $C_2$ ($w_2 = 0.2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega \mathbf{h}_1$ : {(0.3, 0.4, 0.3, 0.4, 0.5)}</td>
<td>{(0.7, 0.8, 0.35, 0.8, 0.6)}</td>
</tr>
<tr>
<td>$\omega \mathbf{h}_2$ : {(0.3, 0.4, 0.5, 0.25, 0.5)}</td>
<td>{(0.5, 0.6, 0.5, 0.6, 0.6)}</td>
</tr>
<tr>
<td>$\omega \mathbf{h}_3$ : {(0.4, 0.5, 0.5, 0.5, 0.6)}</td>
<td>{(0.55, 0.6, 0.75, 0.6, 0.5)}</td>
</tr>
<tr>
<td>$\omega \mathbf{h}_4$ : {(0.3, 0.2, 0.2, 0.3, 0.1)}</td>
<td>{(0.6, 0.5, 0.7, 0.5, 0.5)}</td>
</tr>
<tr>
<td>$\omega \mathbf{h}_5$ : {(0.3, 0.35, 0.6, 0.2, 0.2)}</td>
<td>{(0.6, 0.8, 0.5, 0.45, 0.6)}</td>
</tr>
</tbody>
</table>

Table 2. The mean of weighted extended hesitant fuzzy matrix $\bar{H} = [\bar{h}_{ij}]_{5 \times 4}$.

Step 2. The weighted correlation coefficients between an alternative $A_i$ ($i = 1, 2, 3, 4, 5$) and the ideal alternative $\omega \mathbf{h}_{ideal}$ are computed.

In the experimental results, the arithmetic mean operator, as applied by Zhu and Xu [43] and Lu and Liang [21], and the maximum operator are used.

Step 3. The alternatives are ranked on the basis of their weighted correlation coefficient values, and the maximum value of weighted correlation coefficient is then used to select the best one. All the ranking results of Zhu and Xu [43]’s, Lu and Liang [21]’s and the proposed techniques are respectively shown in Tables 3-6.
Table 3. Zhu and Xu’s [43] generalized expected values (GEVs) for the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$).

Applying the generalized expected values (GEVs), Zhu and Xu [43] concluded that the alternative $A_3$ is the optimal one for being the closest to the ideal values of alternatives with minimum GEV value (Table 3).

<table>
<thead>
<tr>
<th>Ranking order</th>
<th>Energy project $A_1$</th>
<th>Energy project $A_2$</th>
<th>Energy project $A_3$</th>
<th>Energy project $A_4$</th>
<th>Energy project $A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
<td>0.988695</td>
<td>0.971744</td>
<td>0.992133</td>
<td>0.947191</td>
<td>0.967877</td>
</tr>
<tr>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
<td>0.979157</td>
<td>0.94755</td>
<td>0.934583</td>
<td>0.935417</td>
<td>0.93265</td>
</tr>
<tr>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
<td>0.987401</td>
<td>0.971923</td>
<td>0.992084</td>
<td>0.947191</td>
<td>0.967646</td>
</tr>
<tr>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
<td>0.987420</td>
<td>0.971324</td>
<td>0.992082</td>
<td>0.947191</td>
<td>0.967658</td>
</tr>
<tr>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
<td>0.442917</td>
<td>0.4231</td>
<td>0.640208</td>
<td>0.385417</td>
<td>0.526458</td>
</tr>
<tr>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
<td>0.987480</td>
<td>0.971293</td>
<td>0.992084</td>
<td>0.947191</td>
<td>0.967877</td>
</tr>
</tbody>
</table>

Table 4. Lu and Liang’s [21] weighted correlation coefficients between the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$) and the ideal alternative $h_{ideal}$.

<table>
<thead>
<tr>
<th>$\rho_2$ ($= \rho_1 \lambda - 0.5$)</th>
<th>Energy project $A_1$</th>
<th>Energy project $A_2$</th>
<th>Energy project $A_3$</th>
<th>Energy project $A_4$</th>
<th>Energy project $A_5$</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1 \lambda = 0.1$</td>
<td>0.4943</td>
<td>0.4214</td>
<td>0.6429</td>
<td>0.3857</td>
<td>0.5700</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.2$</td>
<td>0.5202</td>
<td>0.4471</td>
<td>0.7049</td>
<td>0.4100</td>
<td>0.5954</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.3$</td>
<td>0.5490</td>
<td>0.4701</td>
<td>0.7284</td>
<td>0.4377</td>
<td>0.6232</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.4$</td>
<td>0.5813</td>
<td>0.5092</td>
<td>0.7537</td>
<td>0.4696</td>
<td>0.6537</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.5$</td>
<td>0.6177</td>
<td>0.5472</td>
<td>0.7808</td>
<td>0.5067</td>
<td>0.6875</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.6$</td>
<td>0.6591</td>
<td>0.5915</td>
<td>0.8101</td>
<td>0.5505</td>
<td>0.7230</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.7$</td>
<td>0.7067</td>
<td>0.6437</td>
<td>0.8416</td>
<td>0.6031</td>
<td>0.7669</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.8$</td>
<td>0.7619</td>
<td>0.7082</td>
<td>0.8762</td>
<td>0.6577</td>
<td>0.8141</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 0.9$</td>
<td>0.8269</td>
<td>0.7823</td>
<td>0.9137</td>
<td>0.7489</td>
<td>0.8676</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>$\rho_1 \lambda = 1$</td>
<td>0.9048</td>
<td>0.8775</td>
<td>0.9548</td>
<td>0.8549</td>
<td>0.9200</td>
<td>$A_3 &gt; A_5 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
</tbody>
</table>

Table 5. Proposed weighted correlation coefficients between the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$) and the ideal alternative $h_{ideal}$ using the arithmetic mean.
The proposed weighted correlation coefficients \( \rho_{V_i} \) \((i = 1, 2, 3, \lambda)\) lead to more reliable decisions than Zhu and Xu’s [43] and Lu and Liang’s [21] ones. This is because the proposed weighted correlation coefficients express correctly the relationship of EHFSs.
Figure 1: Comparison of the proposed weighted correlation coefficients $\rho_{W_i} (i = 1, 2, (3, \lambda))$ given in Tables 5 and 6 with Zhu and Xu [43]’s and Lu and Liang [21]’s ones given respectively in Tables 3 and 4.
In what follows, by keeping in mind the fact that the HFS concept is the special case of EHFS, the interest is to compare the proposed correlation coefficients with some existing correlation coefficients for HFSs.

Adopted originally from [36], we consider a doctor making a proper diagnosis \( D = \{ \text{Viral fever, Malaria, Typhoid, Stomach problem, Chest problem} \} \) for a number of patients \( P = \{ \text{Al, Bob, Joe, Ted} \} \) in accordance with the values of symptoms \( \{ \text{Temperature, Headache, Cough, Stomach pain, Chest pain} \} \). In traditional medical diagnosis is not always realistic to describe the symptoms by the use of crisp values, therefore, such values are assumed to be in the form of HFEs. Table 7 describes the symptom characteristics of diagnoses, while Table 8 presents the symptoms of patients.

<table>
<thead>
<tr>
<th>Symptom</th>
<th>Temperature</th>
<th>Headache</th>
<th>Cough</th>
<th>Stomach pain</th>
<th>Chest pain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viral fever</td>
<td>( b_{VT} = (0.6,0.4,0.3) )</td>
<td>( b_{VH} = (0.7,0.5,0.3,0.2) )</td>
<td>( b_{VCG} = (0.5,0.3) )</td>
<td>( b_{VST} = (0.5,0.4,0.3,0.2,0.1) )</td>
<td>( b_{VCT} = (0.5,0.4,0.2,0.1) )</td>
</tr>
<tr>
<td>Malaria</td>
<td>( b_{MT} = (0.9,0.8,0.7) )</td>
<td>( b_{MH} = (0.5,0.3,0.2,0.1) )</td>
<td>( b_{MCG} = (0.2,0.1) )</td>
<td>( b_{MST} = (0.6,0.5,0.3,0.2,0.1) )</td>
<td>( b_{MCT} = (0.4,0.3,0.2,0.1) )</td>
</tr>
<tr>
<td>Typhoid</td>
<td>( b_{HT} = (0.6,0.3,0.1) )</td>
<td>( b_{HH} = (0.9,0.8,0.7,0.6) )</td>
<td>( b_{HCG} = (0.5,0.3) )</td>
<td>( b_{HST} = (0.5,0.4,0.3,0.2,0.1) )</td>
<td>( b_{HCT} = (0.6,0.4,0.3,0.1) )</td>
</tr>
<tr>
<td>Stomach problem</td>
<td>( b_{ST} = (0.5,0.4,0.2) )</td>
<td>( b_{SH} = (0.4,0.3,0.2,0.1) )</td>
<td>( b_{SCG} = (0.4,0.3) )</td>
<td>( b_{SST} = (0.9,0.8,0.7,0.6,0.5) )</td>
<td>( b_{SCHT} = (0.5,0.4,0.2,0.1) )</td>
</tr>
<tr>
<td>Chest problem</td>
<td>( b_{CT} = (0.3,0.2,0.1) )</td>
<td>( b_{CH} = (0.5,0.3,0.2,0.1) )</td>
<td>( b_{CCG} = (0.3,0.2) )</td>
<td>( b_{CHS} = (0.3,0.4,0.5,0.3,0.2) )</td>
<td>( b_{CCH} = (0.3,0.4,0.5,0.3,0.2) )</td>
</tr>
</tbody>
</table>

Table 7. Symptom characteristics for diagnoses in terms of HFSs.
Table 8. Symptom characteristics for patients in terms of HFSs.

To describe the symptom characteristics using EHFEs, the pessimistic EHFE extension of the HFEs presented in Tables 7 and 8 is assumed, which results in Tables 9 and 10.

The subsequent procedure implements the proposed correlation coefficients for EHFEs between the symptoms characteristic of each diagnose and that of each patient. All these procedures are summarized in Tables 11 to 17. With respect to the given results, it is easily found that

1. The values in Tables 11, 13, 15, 16 and 17 are all in the unit interval [0, 1], meanwhile, there are some negative values in Tables 19-21. Such an outcome shows that the proposed correlation coefficients lead to wider discrimination with values in [−1, 1];

2. The values in Table 11 are all very similar and vary within a narrow range, which makes difficult drawing a realistic conclusion in this case. However, the proposed correlation coefficients provide a wide range of values.

3. All the above mentioned correlation coefficients return the results on the basis of pairwise comparison of HFEs, meanwhile, through the computation of proposed EHFE-based correlation coefficients, the comparison is carried out in overall form. This point can be understood by considering the unification process given in Tables 9 and 10.

<table>
<thead>
<tr>
<th>Viral fever</th>
<th>Temperature</th>
<th>Headache</th>
<th>Cough</th>
<th>Stomach pain</th>
<th>Chest pain</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>b_{VT} = (0.6, 0.4, 0.3, 0.3, 0.3)</td>
<td>b_{VR} = (0.7, 0.5, 0.3, 0.2, 0.2)</td>
<td>b_{VCO} = (0.5, 0.3, 0.3, 0.2, 0.3)</td>
<td>b_{VS} = (0.5, 0.4, 0.3, 0.2, 0.1)</td>
<td>b_{VCCH} = (0.5, 0.4, 0.2, 0.1, 0.1)</td>
</tr>
<tr>
<td>Malaria</td>
<td>b_{MVT} = (0.9, 0.8, 0.7, 0.7, 0.7)</td>
<td>b_{MR} = (0.5, 0.3, 0.2, 0.1, 0.1)</td>
<td>b_{MCO} = (0.5, 0.3, 0.2, 0.1, 0.1)</td>
<td>b_{MS} = (0.5, 0.3, 0.2, 0.1, 0.1)</td>
<td>b_{MCH} = (0.5, 0.3, 0.2, 0.1, 0.1)</td>
</tr>
<tr>
<td>Typhoid</td>
<td>b_{THT} = (0.6, 0.3, 0.1, 0.1, 0.1)</td>
<td>b_{TH} = (0.9, 0.8, 0.7, 0.6, 0.5)</td>
<td>b_{TCO} = (0.5, 0.3, 0.3, 0.3, 0.3)</td>
<td>b_{TS} = (0.5, 0.4, 0.3, 0.2, 0.1)</td>
<td>b_{TCH} = (0.5, 0.4, 0.3, 0.1, 0.1)</td>
</tr>
<tr>
<td>Stomach problem</td>
<td>b_{SHT} = (0.5, 0.4, 0.2, 0.2, 0.2)</td>
<td>b_{SR} = (0.4, 0.3, 0.2, 0.1, 0.1)</td>
<td>b_{SCO} = (0.4, 0.3, 0.3, 0.3, 0.3)</td>
<td>b_{SS} = (0.9, 0.8, 0.7, 0.6, 0.5)</td>
<td>b_{SCH} = (0.5, 0.4, 0.2, 0.1, 0.1)</td>
</tr>
<tr>
<td>Chest problem</td>
<td>b_{CHT} = (0.3, 0.2, 0.1, 0.1, 0.1)</td>
<td>b_{CHR} = (0.5, 0.3, 0.2, 0.1, 0.1)</td>
<td>b_{CCHO} = (0.3, 0.2, 0.2, 0.2, 0.2)</td>
<td>b_{CSS} = (0.7, 0.6, 0.5, 0.3, 0.2)</td>
<td>b_{CCH} = (0.9, 0.8, 0.7, 0.6, 0.6)</td>
</tr>
</tbody>
</table>
Table 9. Symptom characteristics for diagnoses in terms of EHFSs.

<table>
<thead>
<tr>
<th></th>
<th>Temperature</th>
<th>Headache</th>
<th>Cough</th>
<th>Stomach pain</th>
<th>Chest pain</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>At</strong></td>
<td>$h_{AT} = (0.9, 0.7, 0.5, 0.5, 0.5)$</td>
<td>$h_{AH} = (0.4, 0.3, 0.2, 0.1, 0.1)$</td>
<td>$h_{ACO} = (0.4, 0.3, 0.2, 0.1, 0.1)$</td>
<td>$h_{AS} = (0.6, 0.5, 0.4, 0.2, 0.1)$</td>
<td>$h_{ACH} = (0.4, 0.3, 0.2, 0.1, 0.1)$</td>
</tr>
<tr>
<td><strong>Bob</strong></td>
<td>$h_{BT} = (0.5, 0.4, 0.3, 0.2, 0.2)$</td>
<td>$h_{BH} = (0.5, 0.4, 0.3, 0.2, 0.2)$</td>
<td>$h_{BCO} = (0.2, 0.1, 0.1, 0.1, 0.1)$</td>
<td>$h_{BS} = (0.9, 0.8, 0.6, 0.5, 0.4)$</td>
<td>$h_{BCH} = (0.5, 0.4, 0.3, 0.2, 0.2)$</td>
</tr>
<tr>
<td><strong>Joe</strong></td>
<td>$h_{JT} = (0.9, 0.7, 0.6, 0.6, 0.6)$</td>
<td>$h_{JH} = (0.7, 0.6, 0.5, 0.4, 0.3)$</td>
<td>$h_{JCO} = (0.3, 0.2, 0.1, 0.1, 0.1)$</td>
<td>$h_{JS} = (0.6, 0.4, 0.3, 0.2, 0.1)$</td>
<td>$h_{JCH} = (0.6, 0.4, 0.3, 0.2, 0.1, 0.1)$</td>
</tr>
<tr>
<td><strong>Ted</strong></td>
<td>$h_{TT} = (0.8, 0.7, 0.5, 0.5, 0.5)$</td>
<td>$h_{TH} = (0.6, 0.5, 0.4, 0.2, 0.2)$</td>
<td>$h_{TCO} = (0.4, 0.3, 0.2, 0.3, 0.3)$</td>
<td>$h_{TS} = (0.6, 0.4, 0.3, 0.2, 0.1)$</td>
<td>$h_{TCH} = (0.5, 0.4, 0.3, 0.2, 0.1, 0.1)$</td>
</tr>
</tbody>
</table>

Table 10. Symptom characteristics for patients in terms of EHFSs.

In the subsequent tables, the notations $V, M, T, S$ and $C$ refer respectively to Viral fever, Malaria, Typhoid, Stomach problem and Chest problem. Furthermore, we set

\[
\mathcal{H}_V = \{h_{VT}, h_{VH}, h_{VCO}, h_{VS}, h_{VCH}\},
\]
(66)

\[
\mathcal{H}_M = \{h_{MT}, h_{MH}, h_{MCO}, h_{MS}, h_{MCH}\},
\]
(67)

\[
\mathcal{H}_T = \{h_{TT}, h_{TH}, h_{TCO}, h_{TS}, h_{TCH}\},
\]
(68)

\[
\mathcal{H}_S = \{h_{ST}, h_{SH}, h_{SCO}, h_{SS}, h_{SCH}\},
\]
(69)

\[
\mathcal{H}_C = \{h_{CT}, h_{CH}, h_{COC}, h_{CS}, h_{CCH}\},
\]
(70)

and

\[
\mathcal{H}_A = \{h_{AT}, h_{AH}, h_{ACO}, h_{AS}, h_{ACH}\},
\]
(71)

\[
\mathcal{H}_B = \{h_{BT}, h_{BH}, h_{BCO}, h_{BS}, h_{BCH}\},
\]
(72)

\[
\mathcal{H}_J = \{h_{JT}, h_{JH}, h_{JCO}, h_{JS}, h_{JCH}\},
\]
(73)

\[
\mathcal{H}_T = \{h_{TT}, h_{TH}, h_{TCO}, h_{TS}, h_{TCH}\}. 
\]
(74)
<table>
<thead>
<tr>
<th></th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>0.9969</td>
<td>0.9929</td>
<td>0.9800</td>
<td>0.9902</td>
<td>0.9878</td>
<td>V &gt; M &gt; S &gt; C &gt; T</td>
</tr>
<tr>
<td>Bob</td>
<td>0.9900</td>
<td>0.9862</td>
<td>0.9792</td>
<td>0.9921</td>
<td>0.9909</td>
<td>S &gt; C &gt; V &gt; M &gt; T</td>
</tr>
<tr>
<td>Joe</td>
<td>0.9927</td>
<td>0.9877</td>
<td>0.9817</td>
<td>0.9750</td>
<td></td>
<td>M &gt; V &gt; S &gt; C &gt; T</td>
</tr>
<tr>
<td>Ted</td>
<td>0.9942</td>
<td>0.9899</td>
<td>0.9878</td>
<td>0.9772</td>
<td></td>
<td>V &gt; M &gt; S &gt; T &gt; C</td>
</tr>
</tbody>
</table>

Table 11. Correlation coefficient values for each patient to diagnoses based on Xu and Xia’s [36] technique.

<table>
<thead>
<tr>
<th></th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>0.4597</td>
<td>0.9187</td>
<td>-0.4288</td>
<td>0.123</td>
<td>-0.5372</td>
<td>M &gt; V &gt; S &gt; T &gt; C</td>
</tr>
<tr>
<td>Bob</td>
<td>-0.5715</td>
<td>0.2546</td>
<td>-0.3166</td>
<td>0.6074</td>
<td>0.3042</td>
<td>S &gt; C &gt; M &gt; T &gt; V</td>
</tr>
<tr>
<td>Joe</td>
<td>0.5395</td>
<td>0.9803</td>
<td>-0.1217</td>
<td>-0.1017</td>
<td>-0.4836</td>
<td>M &gt; V &gt; S &gt; T &gt; C</td>
</tr>
<tr>
<td>Ted</td>
<td>0.7330</td>
<td>0.9082</td>
<td>0.0210</td>
<td>-0.2230</td>
<td>-0.6596</td>
<td>M &gt; V &gt; T &gt; S &gt; C</td>
</tr>
</tbody>
</table>

Table 12. Correlation coefficient values for each patient to diagnoses based on Liao et al.’s [19] technique.

<table>
<thead>
<tr>
<th></th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>0.4599</td>
<td>0.8246</td>
<td>0.3578</td>
<td>0.6016</td>
<td>0.3383</td>
<td>M &gt; S &gt; V &gt; T &gt; C</td>
</tr>
<tr>
<td>Bob</td>
<td>0.2644</td>
<td>0.7146</td>
<td>0.3377</td>
<td>0.8115</td>
<td>0.7355</td>
<td>S &gt; C &gt; M &gt; V &gt; T</td>
</tr>
<tr>
<td>Joe</td>
<td>0.7990</td>
<td>0.9080</td>
<td>0.2797</td>
<td>0.5329</td>
<td>0.4267</td>
<td>M &gt; V &gt; S &gt; C &gt; T</td>
</tr>
<tr>
<td>Ted</td>
<td>0.7476</td>
<td>0.9057</td>
<td>0.4150</td>
<td>0.5601</td>
<td>0.3861</td>
<td>M &gt; V &gt; S &gt; T &gt; C</td>
</tr>
</tbody>
</table>

Table 13. Correlation coefficient values for each patient to diagnoses based on the synthetic correlation coefficient of Guan et al.’s [16] technique.
Table 14. Correlation coefficient values for each patient to diagnoses based on the mean correlation coefficient of Guan et al.’s [16] technique.

<table>
<thead>
<tr>
<th></th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>$\rho(N_A, N_f) = 0.9322$</td>
<td>$\rho(N_A, N_M) = 0.8966$</td>
<td>$\rho(N_A, N_T) = 0.8914$</td>
<td>$\rho(N_A, N_S) = 0.8808$</td>
<td>$\rho(N_A, N_C) = 0.7283$</td>
<td>$S &gt; M &gt; C &gt; T &gt; V$</td>
</tr>
<tr>
<td>Bob</td>
<td>$\rho(N_B, N_f) = 0.8820$</td>
<td>$\rho(N_B, N_M) = 0.938$</td>
<td>$\rho(N_B, N_T) = 0.8291$</td>
<td>$\rho(N_B, N_S) = 0.9700$</td>
<td>$\rho(N_B, N_C) = 0.8722$</td>
<td>$S &gt; V &gt; C &gt; M &gt; T$</td>
</tr>
<tr>
<td>Joe</td>
<td>$\rho(N_J, N_f) = 0.8420$</td>
<td>$\rho(N_J, N_M) = 0.9787$</td>
<td>$\rho(N_J, N_T) = 0.8486$</td>
<td>$\rho(N_J, N_S) = 0.8460$</td>
<td>$\rho(N_J, N_C) = 0.7415$</td>
<td>$M &gt; V &gt; T &gt; S &gt; C$</td>
</tr>
<tr>
<td>Ted</td>
<td>$\rho(N_T, N_f) = 0.9706$</td>
<td>$\rho(N_T, N_M) = 0.9555$</td>
<td>$\rho(N_T, N_T) = 0.8963$</td>
<td>$\rho(N_T, N_S) = 0.8716$</td>
<td>$\rho(N_T, N_C) = 0.7618$</td>
<td>$V &gt; M &gt; T &gt; S &gt; C$</td>
</tr>
</tbody>
</table>

Table 15. Correlation coefficient values for each patient to diagnoses based on the variance correlation coefficient of Guan et al.’s [16] technique.

<table>
<thead>
<tr>
<th></th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>$\rho(N_A, N_f) = 0.8780$</td>
<td>$\rho(N_A, N_M) = 0.9511$</td>
<td>$\rho(N_A, N_T) = 0.7317$</td>
<td>$\rho(N_A, N_S) = 0.8664$</td>
<td>$\rho(N_A, N_C) = 0.6567$</td>
<td>$M &gt; V &gt; S &gt; T &gt; C$</td>
</tr>
<tr>
<td>Bob</td>
<td>$\rho(N_B, N_f) = 0.8469$</td>
<td>$\rho(N_B, N_M) = 0.8082$</td>
<td>$\rho(N_B, N_T) = 0.7240$</td>
<td>$\rho(N_B, N_S) = 0.9275$</td>
<td>$\rho(N_B, N_C) = 0.7595$</td>
<td>$S &gt; V &gt; M &gt; C &gt; T$</td>
</tr>
<tr>
<td>Joe</td>
<td>$\rho(N_J, N_f) = 0.8284$</td>
<td>$\rho(N_J, N_M) = 0.9401$</td>
<td>$\rho(N_J, N_T) = 0.8993$</td>
<td>$\rho(N_J, N_S) = 0.8108$</td>
<td>$\rho(N_J, N_C) = 0.7050$</td>
<td>$M &gt; V &gt; S &gt; T &gt; C$</td>
</tr>
<tr>
<td>Ted</td>
<td>$\rho(N_T, N_f) = 0.8661$</td>
<td>$\rho(N_T, N_M) = 0.9313$</td>
<td>$\rho(N_T, N_T) = 0.8422$</td>
<td>$\rho(N_T, N_S) = 0.8471$</td>
<td>$\rho(N_T, N_C) = 0.7138$</td>
<td>$M &gt; V &gt; S &gt; T &gt; C$</td>
</tr>
</tbody>
</table>

Table 16. Correlation coefficient values for each patient to diagnoses based on Eq. (14) of Chen et al.’s [2] technique.

Table 17. Correlation coefficient values for each patient to diagnoses based on Eq. (15) of Chen et al.’s [2] technique.
In order to compare the performance of the above-mentioned correlation coefficients for each patient to the possible diagnosis with that of the EHFE proposed correlation coefficients, we here aggregate all the ranking orders in Tables 11 to 17 following the initial steps of Farhadinia and Herrera-Viedma’s [12] algorithm. The implementation of the collective majority decision mapping [12] enables us to extract the collective ranking order as the following.

<table>
<thead>
<tr>
<th>Collective ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
</tr>
<tr>
<td>Bob</td>
</tr>
<tr>
<td>Joe</td>
</tr>
<tr>
<td>Ted</td>
</tr>
</tbody>
</table>

Table 18. The collective ranking order of correlation coefficients values given in Tables 11 to 17.

Now, if we apply the proposed EHFE correlation coefficients between the symptoms characteristic of each diagnose and that of each patient, then the results will be as observed in Tables 19 and 21.

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>$\rho_{M_A, V_1} = 0.4444$</td>
<td>$\rho_{M_A, N_{M_A}} = 0.6826$</td>
<td>$\rho_{M_A, N_C} = -0.3975$</td>
<td>$\rho_{M_A, N_S} = 0.2235$</td>
<td>$\rho_{M_A, N_T} = -0.3873$</td>
<td>$M &gt; V &gt; S &gt; T &gt; C$</td>
<td></td>
</tr>
<tr>
<td>Bob</td>
<td>$\rho_{M_B, V_1} = 0.0561$</td>
<td>$\rho_{M_B, N_{M_B}} = 0.2052$</td>
<td>$\rho_{M_B, N_C} = -0.1122$</td>
<td>$\rho_{M_B, N_S} = 0.8346$</td>
<td>$\rho_{M_B, N_T} = 0.4336$</td>
<td>$S &gt; C &gt; M &gt; V &gt; T$</td>
<td></td>
</tr>
<tr>
<td>Joe</td>
<td>$\rho_{M_J, V_1} = 0.0702$</td>
<td>$\rho_{M_J, N_{M_J}} = 0.0510$</td>
<td>$\rho_{M_J, N_C} = -0.1586$</td>
<td>$\rho_{M_J, N_S} = 0.0099$</td>
<td>$\rho_{M_J, N_T} = -0.3273$</td>
<td>$M &gt; V &gt; S &gt; T &gt; C$</td>
<td></td>
</tr>
<tr>
<td>Ted</td>
<td>$\rho_{M_T, V_1} = 0.0942$</td>
<td>$\rho_{M_T, N_{M_T}} = 0.8752$</td>
<td>$\rho_{M_T, N_C} = -0.0232$</td>
<td>$\rho_{M_T, N_S} = -0.0469$</td>
<td>$\rho_{M_T, N_T} = -0.5015$</td>
<td>$M &gt; V &gt; T &gt; S &gt; C$</td>
<td></td>
</tr>
</tbody>
</table>

Table 19. Correlation coefficient values for each patient to diagnoses based on the proposed $\rho_1$-based technique.
Table 20. Correlation coefficient values for each patient to diagnoses based on the proposed $\rho_2$-based technique.

<table>
<thead>
<tr>
<th>$\rho_2$ (Table 18)</th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_2(A, A)$</td>
<td>$\rho_{NA, NA} = 0.4300$</td>
<td>$\rho_{NA, NA} = 0.2651$</td>
<td>$\rho_{NA, NA} = 0.1505$</td>
<td>$\rho_{NA, NA} = 0.0857$</td>
<td>$\rho_{NA, NA} = -0.1247$</td>
<td>$V &gt; M &gt; S &gt; C &gt; T$</td>
</tr>
<tr>
<td>Bob</td>
<td>$\rho_{NB, NB} = -0.0480$</td>
<td>$\rho_{NB, NB} = 0.0704$</td>
<td>$\rho_{NB, NB} = -0.0376$</td>
<td>$\rho_{NB, NB} = 0.3827$</td>
<td>$\rho_{NB, NB} = 0.1234$</td>
<td>$S &gt; C &gt; M &gt; T &gt; V$</td>
</tr>
<tr>
<td>Joe</td>
<td>$\rho_{NJ, NJ} = 0.4001$</td>
<td>$\rho_{NJ, NJ} = 0.2564$</td>
<td>$\rho_{NJ, NJ} = -0.0668$</td>
<td>$\rho_{NJ, NJ} = 0.0034$</td>
<td>$\rho_{NJ, NJ} = -0.0936$</td>
<td>$V &gt; M &gt; S &gt; T &gt; C$</td>
</tr>
<tr>
<td>Ted</td>
<td>$\rho_{NP, NP} = 0.8062$</td>
<td>$\rho_{NP, NP} = 0.3156$</td>
<td>$\rho_{NP, NP} = -0.0105$</td>
<td>$\rho_{NP, NP} = -0.0216$</td>
<td>$\rho_{NP, NP} = -0.1938$</td>
<td>$V &gt; M &gt; T &gt; S &gt; C$</td>
</tr>
</tbody>
</table>

Table 21. Correlation coefficient values for each patient to diagnoses based on the proposed $\rho_2$-based technique.

<table>
<thead>
<tr>
<th>$\rho_2$ (Table 19)</th>
<th>Viral fever</th>
<th>Malaria</th>
<th>Typhoid</th>
<th>Stomach problem</th>
<th>Chest problem</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_2(A, A)$</td>
<td>$\rho_{NA, NA} = 0.3115$</td>
<td>$\rho_{NA, NA} = 0.0036$</td>
<td>$\rho_{NA, NA} = -0.2739$</td>
<td>$\rho_{NA, NA} = 0.1540$</td>
<td>$\rho_{NA, NA} = -0.2655$</td>
<td>$M &gt; V &gt; S &gt; C &gt; T$</td>
</tr>
<tr>
<td>Bob</td>
<td>$\rho_{NB, NB} = -0.0393$</td>
<td>$\rho_{NB, NB} = 0.1805$</td>
<td>$\rho_{NB, NB} = -0.0770$</td>
<td>$\rho_{NB, NB} = 0.5731$</td>
<td>$\rho_{NB, NB} = 0.2960$</td>
<td>$S &gt; C &gt; M &gt; V &gt; T$</td>
</tr>
<tr>
<td>Joe</td>
<td>$\rho_{NJ, NJ} = 0.3991$</td>
<td>$\rho_{NJ, NJ} = 0.6543$</td>
<td>$\rho_{NJ, NJ} = -0.1365$</td>
<td>$\rho_{NJ, NJ} = 0.0068$</td>
<td>$\rho_{NJ, NJ} = -0.2235$</td>
<td>$M &gt; V &gt; S &gt; T &gt; C$</td>
</tr>
<tr>
<td>Ted</td>
<td>$\rho_{NP, NP} = 0.4875$</td>
<td>$\rho_{NP, NP} = 0.0022$</td>
<td>$\rho_{NP, NP} = -0.0161$</td>
<td>$\rho_{NP, NP} = -0.0325$</td>
<td>$\rho_{NP, NP} = -0.3457$</td>
<td>$M &gt; V &gt; T &gt; S &gt; C$</td>
</tr>
</tbody>
</table>

The comparison of the collective ranking orders of Table 18 together with that of Tables 19-21 demonstrates that the results of the proposed correlation coefficients for EHFEs are much more like the collective ranking orders of the existing ones, with the minor difference being due to the above-mentioned superiorities of the proposed correlation coefficients over the existing ones (see items 1-3 above).
4.3 Application of correlation coefficients between EHFSs to the supplier selection problem

Here, we adopt from [22, 37] an actual problem in which an automotive company aims to determine the most beneficial supplier for one of key procedure of manufacturing elements. Four possible suppliers are evaluated using the following attributes:

- $C_1$: Product quality;
- $C_2$: Relationship closeness;
- $C_3$: Delivery performance;
- $C_4$: Price,

where the first three attributes are benefit attribute, and the last attribute is a cost attribute.

In order to select the best supplier, the HFSs given in Table 22 (adapted from [22, 37]) are used.

<table>
<thead>
<tr>
<th>Criteria $C_1$</th>
<th>Criteria $C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$h_{11} = {0.2, 0.4, 0.7}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$h_{21} = {0.4, 0.6, 0.7}$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$h_{31} = {0.2, 0.3, 0.6}$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$h_{41} = {0.2, 0.3, 0.5}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Criteria $C_3$</th>
<th>Criteria $C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$h_{13} = {0.2, 0.3, 0.5, 0.7, 0.8}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$h_{23} = {0.3, 0.4, 0.6, 0.8, 0.9}$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$h_{33} = {0.2, 0.4, 0.6, 0.7, 0.8}$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$h_{43} = {0.4, 0.6, 0.7, 0.8, 0.9}$</td>
</tr>
</tbody>
</table>

Table 22. The hesitant fuzzy decision matrix $H = [h_{ij}]_{4 \times 4}$. 

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Following the normalization process of hesitant fuzzy decision matrix $H = [h_{ij}]_{4 \times 4}$ proposed by Lin et al. [22], the normalized hesitant fuzzy matrix with all elements having the same length are derived (see Table 23).

<table>
<thead>
<tr>
<th>Criteria $C_1$</th>
<th>Criteria $C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ $h_{11} = (0.2, 0.2, 0.2, 0.4, 0.7)$</td>
<td>$h_{12} = (0.1, 0.1, 0.2, 0.5, 0.7)$</td>
</tr>
<tr>
<td>$A_2$ $h_{21} = (0.4, 0.4, 0.4, 0.6, 0.7)$</td>
<td>$h_{22} = (0.1, 0.1, 0.2, 0.4, 0.6)$</td>
</tr>
<tr>
<td>$A_3$ $h_{31} = (0.2, 0.2, 0.2, 0.3, 0.6)$</td>
<td>$h_{32} = (0.3, 0.3, 0.4, 0.5, 0.9)$</td>
</tr>
<tr>
<td>$A_4$ $h_{41} = (0.2, 0.2, 0.2, 0.3, 0.5)$</td>
<td>$h_{42} = (0.2, 0.2, 0.3, 0.5, 0.7)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Criteria $C_3$</th>
<th>Criteria $C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ $h_{13} = (0.2, 0.3, 0.5, 0.7, 0.8)$</td>
<td>$h_{14} = (0.1, 0.1, 0.1, 0.4, 0.6)$</td>
</tr>
<tr>
<td>$A_2$ $h_{23} = (0.3, 0.4, 0.6, 0.8, 0.9)$</td>
<td>$h_{24} = (0.1, 0.1, 0.1, 0.2, 0.4)$</td>
</tr>
<tr>
<td>$A_3$ $h_{33} = (0.2, 0.4, 0.6, 0.7, 0.8)$</td>
<td>$h_{34} = (0.3, 0.3, 0.3, 0.4, 0.8)$</td>
</tr>
<tr>
<td>$A_4$ $h_{43} = (0.4, 0.6, 0.7, 0.8, 0.9)$</td>
<td>$h_{44} = (0.1, 0.1, 0.1, 0.2, 0.7)$</td>
</tr>
</tbody>
</table>

**Table 23.** The extended hesitant fuzzy decision matrix $^{w}H = [^{w}h_{ij}]_{4 \times 4}$.

Needless to say that all the entries of Table 23 are indeed in the form of EHFEs, and Table 23 stands for an extended hesitant fuzzy decision matrix. Since all the relative weights of criteria are unknown, similar to Lin et al. [22]'s procedure, the following non-linear optimization problem whose solution determines the optimal weights of criteria as $W^* = (W^*_1, ..., W^*_N)$ is implemented.

$$
\min \sum_{j=1}^{N} [W_j - \frac{\sum_{i=1}^{n} \rho W_1 (h_{ij}, g^{(j)}_i)}{\sum_{j=1}^{N} \sum_{i=1}^{n} \rho W_2 (h_{ij}, g^{(j)}_i)}] ^2
$$

$$
s.t. \sum_{j=1}^{N} W_j = 1,
0 \leq W_j \leq 1, \quad j = 1, ..., N.
$$

In the above setting, the notation $\rho W_1$ stands for the proposed correlation measures $\rho W_1$, $\rho W_2$ and $\rho W_3, \lambda = \frac{1}{2}$.
between the criteria $C_j$ and the overall assessment value of alternatives,

\[ g_i^{(j)} = \oplus_{j=1,j\neq j}^N (w_j h_{i,j}) \]

\[ = \oplus_{j=1,j\neq j}^N \bigcup_{(\gamma_i^{(a_1)}(x_j),...,\gamma_i^{(a_n)}(x_j)) \in h_i(x_j)} \{ (1 - \prod_{j=1,j\neq j}^n [1 - \gamma_i^{(a_j)}(x_j)]^{w_j}) \cdots (1 - \prod_{j=1,j\neq j}^N [1 - \gamma_i^{(a_j)}(x_j)]^{w_j}) \}, \]

\[ i = 1, \ldots, n, \]

and

\[ \varsigma_j = \left[ \sum_{i=1}^n (h_{ij} - E(h_{ij}))^2 \right]^\frac{1}{2}, \quad j = 1, \ldots, N. \]

It should be noted that the optimal weights of criteria $W^* = (W_1^*, \ldots, W_N^*)$ describes the evaluation of alternatives, which is used to determine the ranking order of alternatives.

By considering the cosine similarity between two length-unified HFEs $h_1 = \{ h_1^{(\delta)} \mid \delta = 1, \ldots, l_h \}$ and $h_2 = \{ h_2^{(\delta)} \mid \delta = 1, \ldots, l_h \}$ in the form of

\[ C_{\text{cosine}} (h_1, h_2) = \frac{\sum_{\delta=1}^{l_h} h_1^{(\delta)} \times h_2^{(\delta)}}{(\sum_{\delta=1}^{l_h} (h_1^{(\delta)})^2) \times (\sum_{\delta=1}^{l_h} (h_2^{(\delta)})^2)^\frac{1}{2}}, \]

and the normalized data given in Table 23, Lin et al. [22] obtained the following priority

\[ W_{\text{LWX}}^* = (W_1^*, W_2^*, W_3^*, W_4^*) = (0.2083, 0.1821, 0.2852, 0.3238). \]

For the above-mentioned data in Table 22, Xu and Xia [37] obtained respectively the following relative weights using two methods based on entropy and maximizing deviation:

\[ W_{\text{XX1}}^* = (W_1^*, W_2^*, W_3^*, W_4^*) = (0.1957, 0.2013, 0.1611, 0.4419); \]

\[ W_{\text{XX2}}^* = (W_1^*, W_2^*, W_3^*, W_4^*) = (0.2629, 0.2229, 0.2057, 0.3086). \]

In order to compare the outcomes of proposed correlation measures $\rho_{w_1}$, $\rho_{w_2}$, $\rho_{w_3, \lambda=\frac{1}{3}}$ and those proposed by Lin et al. [22] and Xu and Xia [37], the procedure of Lin et al. [22] given earlier is used to the extended hesitant fuzzy decision matrix $^{\text{eh}} H = [h_{ij}]_{4 \times 4}$ given by Table 23. The outcomes of correlation measures $\rho_{w_1}$, $\rho_{w_2}$ and
Figure 2: Comparison of Lin et al. [22]’s, Xu and Xia [37]’s and the proposed correlation coefficient values.

ρ_{W_{3,\lambda=\frac{1}{2}}} are:

W^{*}_{\rho_{W_{1}}} = (W_{1}^{*}, W_{2}^{*}, W_{3}^{*}, W_{4}^{*}) = (0.1244, 0.0683, 0.3643, 0.4430);

W^{*}_{\rho_{W_{2}}} = (W_{1}^{*}, W_{2}^{*}, W_{3}^{*}, W_{4}^{*}) = (0.1377, 0.1048, 0.3523, 0.4052);

W^{*}_{\rho_{W_{3,\lambda=\frac{1}{2}}}} = (W_{1}^{*}, W_{2}^{*}, W_{3}^{*}, W_{4}^{*}) = (0.2674, 0.1512, 0.2451, 0.3363).

A visual representation of the results obtained is given in Figure 12. It can be easily seen that W_{4} has the largest value according to all considered correlation coefficients, while, the other rest of weights have different order positions, with weight W_{3} being generally greater than weight W_{2}, although this is not in accordance with Xu and Xia [37]’s ranking orders. It is noticed that the results obtained with the proposed correlation coefficients are consistent with those of Lin et al. [22], which support the validity of the proposed correlation coefficients.
5 Conclusions and future works

This article proposed a new class of weighted correlation coefficients of weighted extended hesitant fuzzy sets (WEHFSs) whose main characteristic being the range \([-1, 1]\), which is consistent with this concept in classic statistics. This is the first class of correlation coefficients introduced between WEHFSs with a range different to the unit interval \([0, 1]\). In analysis of three decision making problems under weighted extended hesitant fuzzy environment, served to compare the proposed correlation coefficients with the existing ones, which demonstrated their efficiency and effectiveness. The contribution of this article is expected to be used, in future, in many application fields including sources location, information retrieval, investment decision-making, data mining, equipment evaluation, etc.

Compliance with Ethical Standards

Conflict of Interest

The author declares that he has no conflict of interest.

Human and Animal Rights

This article does not contain any studies with human participants or animals performed by the author.

References


Figure 3: Correlation coefficient values based on Xu and Xia’s [36] technique.
Figure 4: Correlation coefficient values based on Liao et al.’s [19] technique.
Figure 5: Synthetic correlation coefficient values based on Guan et al.’s [16] technique.
Figure 6: Mean correlation coefficient values based on Guan et al.'s [16] technique.
Figure 7: Variance correlation coefficient values based on Guan et al.’s [16] technique.
Figure 8: Correlation coefficient values based on Eq. (14) of Chen et al.’s [2] technique.
Figure 9: Correlation coefficient values based on Eq. (15) of Chen et al.’s [2] technique.
Figure 10: Correlation coefficient values based on the $\rho_1$-based proposed technique.
Figure 11: Correlation coefficient values based on the $\rho_2$-based proposed technique.
Figure 12: Correlation coefficient values based on the $p_3$-based proposed technique.