A note on the internal consistency
of various preference representations

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Abstract

In [3] we presented a fuzzy multipurpose decision making model integrating different preference representations: preference orderings, utility functions and fuzzy preference relations. We complete the decision model studying its internal consistency.

Keywords: Multipurpose decision making, preference orderings, utility functions, fuzzy preference relations, consistency.

1 Introduction

The objective of a decision making process is to classify the alternatives \( X = \{x_1, x_2, \ldots, x_n\}, (n \geq 2) \) from best to worst, using the information about them according to a set of general purposes (experts or criteria) \( E = \{e_1, e_2, \ldots, e_m\}, (m \geq 2) \). In [3] we presented a multipurpose decision making (MPDM) model assuming that the experts’ preferences can be provided in any of the following three preference representations:

1. Preference ordering of the alternatives: \( O^k = (o^k(1), \ldots, o^k(n)) \), where \( o^k(\cdot) \) is a permutation function over the index set, \( \{1, \ldots, n\} \), for the expert, \( e_k \).

2. Fuzzy preference relation: \( P^k \subset X \times X \), with membership function, \( \mu_{p^k} : X \times X \rightarrow [0, 1] \), where \( \mu_{p^k}(x_i, x_j) = p^k_{ij} \) denotes the preference degree or intensity of the alternative \( x_i \) over \( x_j \), and being \( P^k \) assumed additive reciprocal, i.e., \( p^k_{ij} + p^k_{ji} = 1 \).

3. Utility function: \( U^k = \{u^k_i, i = 1, \ldots, n\}, u^k_i \in [0, 1] \), where \( u^k_i \) represents the utility evaluation given by the expert \( e_k \) to the alternative \( x_i \).

This decision model obtains the set of solution alternatives in the following steps:

1. Make the information uniform by means of the transformation function defined in Proposition 1, which assumes the fuzzy preference relations as the base element to make the preferences uniform.

**Proposition 1.** Let \( X \) be a set of alternatives and \( \lambda^k_i \) represents an evaluation of alternative \( x_i \) indicating the performance of \( x_i \) according to a purpose \( e_k \). Then, the intensity of preference of alternative \( x_i \) over alternative \( x_j \), \( p^k_{ij} \) for \( e_k \) is given by the following transformation function \( p^k_{ij} = \varphi(\lambda^k_i, \lambda^k_j) = \frac{1}{2} \cdot [1 + \psi(\lambda^k_i, \lambda^k_j) - \psi(\lambda^k_j, \lambda^k_i)] \), where \( \psi \) is a function verifying (i) \( \psi(z, z) = \frac{1}{2}, \forall z \in \mathbb{R} \), and (ii) \( \psi \) is non decreasing in the first argument and non increasing in the second argument.
Corollary 1.1. If \( \lambda^k_i = \phi^k(i) \), and \( \psi(\lambda^k_i, \lambda^k_j) = F(\lambda^k_j - \lambda^k_i) \), where \( F \) is any non-decreasing function, then \( \varphi \) transforms preference orderings into fuzzy preference relations.

Corollary 1.2. If \( \lambda^k_i = u^k_i \) and \( \psi(z, y) = \begin{cases} \frac{s(z)}{2} & \text{if } (z, y) \neq (0, 0), \\ 0 & \text{if } (z, y) = (0, 0) \end{cases} \), where \( s : [0, 1] \rightarrow \mathbb{R}^+ \) is a non-decreasing and continuous function, verifying \( s(0) = 0 \), then \( \varphi \) transforms preference orderings into fuzzy preference relations.

2. Aggregate the individual fuzzy preference relations \( \{P^1, \ldots, P^n\} \) in a collective fuzzy preference relation \( P^c \) by means of a linguistic quantifier guided OWA operator \( \phi_Q \) [11].

3. Exploit \( P^c \) by means of following choice degrees of alternatives [2]:

(a) Quantifier guided dominance degree, which is defined for the alternative, \( x_i \), as \( \text{QGDD}_i = \phi_Q(p^*_i, j = 1, \ldots, n, j \neq i) \).

(b) Quantifier guided non dominance degree, which is defined for the alternative, \( x_i \), as \( \text{QGNDD}_i = \phi_Q(1 - p^*_i, j = 1, \ldots, n, j \neq i), p^*_i = \max\{p^*_{ij} - p^*_{ij}, 0\} \).

In Section 2 we study the internal consistency of this decision model by analyzing the consistency of the transformation function per each expert. Finally, we point out some concluding remarks.

2 The consistency of transformation function \( \varphi \)

In this section, we demonstrate that the transformation function proposed in Proposition 1 acts coherently according to both choice degrees of alternatives (QGDD, QGNDD), because the ranking among the alternatives that we can obtain from any of the considered preference representations (preference ordering and utility values) is not disturbed if we apply any of the two choice degrees on the respective fuzzy preference relation obtained via a transformation function. Previously, we are going to present some interesting consequences followed from Proposition 1, and necessary to prove the consistency of \( \varphi \).

Corollary 1.3. Suppose a set of evaluations, \( \{\lambda^k_1, \ldots, \lambda^k_n\} \), provided on \( X \) by an expert \( e_k \). Without loss of generality, assume the following order, \( 0 < \lambda^k_i \leq \ldots \leq \lambda^k_n \leq 1 \). Then, the following restriction is verified: \( p^k_{i1} \geq \ldots \geq p^k_{i(n-1)} \geq p^k_{in} = \frac{1}{2} \geq p^k_{i(n+1)} \geq \ldots \geq p^k_{in} \).

Proof. Let \( i, j, s \in \{1, \ldots, n\} \) be such that \( s < i < j \). Then, we have \( 0 < \lambda^k_s \leq \lambda^k_i \leq \lambda^k_j \leq 1 \), and therefore from Proposition 1

\[
\psi(\lambda^k_i, \lambda^k_j) - \psi(\lambda^k_s, \lambda^k_i) \leq \frac{1}{2} \cdot \left\{ \left[ \psi(\lambda^k_i, \lambda^k_j) - \psi(\lambda^k_s, \lambda^k_i) \right] + \left[ \psi(\lambda^k_s, \lambda^k_i) - \psi(\lambda^k_s, \lambda^k_j) \right] \right\} =
\]

\[
\frac{1}{2} \cdot \left[ 1 + \psi(\lambda^k_i, \lambda^k_j) - \psi(\lambda^k_s, \lambda^k_j) \right] - \left[ 1 + \psi(\lambda^k_s, \lambda^k_i) - \psi(\lambda^k_s, \lambda^k_j) \right] = p^k_{ij} - p^k_{is} \leq 0,
\]

which implies: \( p^k_{i1} \geq \ldots \geq p^k_{i(n-1)} \geq p^k_{in} = \frac{1}{2} \geq p^k_{i(n+1)} \geq \ldots \geq p^k_{in} \).

Corollary 1.4. Suppose a set of evaluations \( \{\lambda^k_1, \ldots, \lambda^k_n\} \) provided on \( X \) by an expert \( e_k \). Without loss of generality, assume the following order, \( 0 < \lambda^k_i \leq \ldots \leq \lambda^k_n \leq 1 \). \( \forall i, j, s \in \{1, \ldots, n\} \) such that \( i < j \), then \( p^k_{js} \geq p^k_{is} \).
Proof. From Proposition 1 \( \psi(\lambda_i^k, \lambda_j^k) \leq \psi(\lambda_i^k, \lambda_j^k) \), and therefore

\[
p_{ts}^k - p_{js}^k = \frac{1}{2} \cdot \left\{ \left[ 1 + \psi(\lambda_i^k, \lambda_j^k) - \psi(\lambda_s^k, \lambda_j^k) \right] - \left[ 1 + \psi(\lambda_i^k, \lambda_j^k) - \psi(\lambda_s^k, \lambda_j^k) \right] \right\}
\]

\[
= \frac{1}{2} \cdot \left\{ \psi(\lambda_i^k, \lambda_j^k) - \psi(\lambda_s^k, \lambda_j^k) \right\} \leq 0
\]

what implies \( p_{js}^k \geq p_{ts}^k \), \( \forall i, j, s \), such that \( i < j \).

Proposition 2. Let \( i, j \in \{1, ..., n\} \) be such that \( i < j \), assuming the evaluations given by an expert \( e_k \) verify \( \lambda_i^k \leq \lambda_j^k \), then the dominance and non dominance choice degrees obtained from the fuzzy preference relation \( P_k \) satisfy the following relationships: (i) \( QGDD_j^k \geq QGDD_i^k \), and (ii) \( QGNDD_j^k \geq QGNDD_i^k \).

Proof. Firstly, we prove that

\[
QGDD_j^k \geq QGDD_i^k \quad \forall i, j.
\]

Using Collorary 1.4 we know that if \( i < j \), then \( p_{jt}^k \geq p_{it}^k \), \( \forall t \), and particularly, \( \forall t \in \{1, ..., i - 1\} \cup \{j + 1, ..., n\} \). On the other hand, using Collorary 1.3 we have that \( p_{it}^k \geq p_{it+1}^k \), \( \forall i, t \), and therefore \( \forall i < j \) we have that

\[
p_{jt}^k \geq p_{it}^k \geq p_{it+1}^k \quad \forall t,
\]

and particularly, \( \forall t \in \{i, ..., j - 1\} \). Therefore, concluding, the following relationship is satisfied,

\[
QGDD_j^k = \sum_{t=1}^{n} w_t \cdot p_{jt}^k \geq \sum_{t=1}^{n} w_t \cdot p_{it}^k = QGDD_i^k,
\]

with \( w_t \) being the weights used in the OWA operator applied to obtain the degrees \( QGDD_j^k \) and \( QGDD_i^k \). Secondly, we prove that

\[
QGNDD_j^k \geq QGNDD_i^k \quad \forall i, j.
\]

Using Collorary 1.4 we know that if \( i < j \), then \( p_{jt}^k \geq p_{it}^k \), \( \forall t \). Using Collorary 1.3 we know that if \( i < j \), then \( p_{it}^k \geq p_{jt}^k \), \( \forall t \). Therefore, the following expression is satisfied

\[
p_{it}^{k,s} = \max\{p_{it}^k - p_{it}^s, 0\} \geq \max\{p_{jt}^k - p_{jt}^s, 0\} = p_{jt}^{k,s}, \quad \forall t.
\]

Then, particularly we have

\[
1 - p_{it}^{k,s} \leq 1 - p_{jt}^{k,s} \quad \forall t \in \{1, ..., i - 1\} \cup \{j + 1, ..., n\}.
\]

On the other hand, using Collorary 1.4 we know that \( p_{it+1}^k \geq p_{it}^k \), \( \forall t \). Using Collorary 1.3 we know that \( p_{it}^k \geq p_{it+1}^k \), \( \forall t \). Therefore, the following expression is satisfied

\[
p_{it+1}^{k,s} = \max\{p_{it+1}^k - p_{it}^k, 0\} \geq \max\{p_{it}^k - p_{it}^s, 0\} = p_{it}^{k,s}, \quad \forall t.
\]

Then, particularly we have \( 1 - p_{it}^{k,s} \geq 1 - p_{jt}^{k,s} \), \( \forall t \in \{i, ..., j - 1\} \), and therefore \( \forall i < j \) we have that \( 1 - p_{jt}^{k,s} \geq 1 - p_{jt}^{k,s} \), \( \forall t \in \{i, ..., j - 1\} \). Concluding, the following relationship is satisfied

\[
QGNDD_j^k = \sum_{t=1}^{n} w_t \cdot (1 - p_{jt}^{k,s}) \geq \sum_{t=1}^{n} w_t \cdot (1 - p_{it}^{k,s}) = QGNDD_i^k.
\]
3 Concluding remarks

Proposition 2 guarantees the internal consistency of the decision process presented in [3]. We must remark that the issue is valid even if the elements of the main diagonal of the fuzzy preference relations are used in the calculus of the dominance and non dominance degrees and independent of the linguistic quantifiers used.

Finally, we should point out that the decision model presented in [3] is mainly used in Social Theory [4] and outranking methods [7, 8]. Therefore, there exist

(i) other proposals on transformation functions, e.g. those based on the concept of preference structures (strict preference, indifference and incomparability) [5, 6, 8], and also

(ii) other proposals on choice degrees, e.g. those based on scoring functions (the leaving flow, entering flow the net flow, etc.) [1, 6, 8, 9].

In consequence, a comparative study of our decision model with these ones similar to that done in [10] may be an interesting future research.

References


