Transformational Programming

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Waterfall Model

1. **Requirements Elicitation:** Analyse the problem domain and determine what the program is required to do
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3. **Implementation:** Write source code to implement the design in a particular programming language

4. **Verification:** Run tests and debugging

5. **Maintenance:** Any modifications required after delivery to correct faults, improve performance, or adapt the product to a modified environment
Waterfall Model

Requirements → Design → Implementation → Verification → Maintenance
Proving Correctness

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Program testing can be used to show the presence of bugs, but never to show their absence — Dijkstra 1970.

To prove that a program is correct we need two things:

1. A precise mathematical specification which defines what the program is supposed to do, and

2. A mathematical proof that the program satisfies the specification
Program Verification

- Program code
- Pre/postconditions
- Loop invariants
- Verification
Verification of a Loop

1. Determine the loop termination condition;
2. Determine the loop body;
3. Determine a suitable loop invariant;
4. Prove that the loop invariant is preserved by the loop body;
5. Determine a variant function for the loop;
6. Prove that the variant function is reduced by the loop body (thereby proving termination of the loop);
7. Prove that the combination of the invariant plus the termination condition satisfies the specification for the loop.
Dijkstra’s Approach

- Pre/postconditions
- Program code
- Loop Invariants
- Verification
Invariant Based Programming

- Pre/postconditions
- Loop invariants
- Program code
- Verification
Common Factors

- All these development methods require the invention of *loop invariants*.
- In all these methods, the final step is *Verification*.
- The program under development is not guaranteed to be correct until verification is complete.
- Introducing a loop requires developing a loop invariant and variant expression.
Transformational Programming

- Formal Program Specification
- Informal Implementation ideas

Program 1

Program n
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Transformational Programming Stages

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5. **Recursion Removal**: Apply the Generic Recursion Removal Theorem.
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4. **Recursion Introduction**: Apply the Recursive Implementation Theorem to produce a recursive program with no remaining copies of the specification.

5. **Recursion Removal**: Apply the Generic Recursion Removal Theorem.

6. **Optimisation**: As required.
A formal specification defines precisely what the program is required to accomplish, without necessarily giving any indication as to how the task is to be accomplished.

A formal specification for a factorial program could be written as:

\[ r := n! \]

A formal specification for the Quicksort algorithm for sorting the array \(A[a \ldots b]\) is the statement \(\text{SORT}(a, b)\):

\[ A[a \ldots b] := A'[a \ldots b].(\text{sorted}(A'[a \ldots b]) \land \text{permutation}(A[a \ldots b], A'[a \ldots b])) \]
The form of the specification should mirror the real-world nature of the requirements. Construct suitable abstractions such that local changes to the requirements involve local changes to the specification.

The notation used for the specification should permit unambiguous expression of requirements and support rigorous analysis to uncover contradictions and omissions.
The *Elaboration* stage is the process of applying transformations to take out the simple cases.

Typically, this uses Splitting a Tautology to duplicate the specification, then insert assertions, then use the assertions to refine the appropriate copy of the specification to the trivial implementation.

For the factorial program, the simplest case is $0! = 1$, so we split on the test $n = 0$:

```
if $n = 0$ then $r := n!$ else $r := n!$ fi
```

and simplify the case where $n = 0$:

```
if $n = 0$ then $r := 1$ else $r := n!$ fi
```
Elaboration

For the sort function, the simplest case is when $a \geq b$. In this case, the array has zero or one elements and is therefore already sorted.
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SORT($a, b$) is transformed to:

```plaintext
if $a \geq b$ then SORT($a, b$)
    else SORT($a, b$) fi
```
Elaboration

For the sort function, the simplest case is when $a \geq b$. In this case, the array has zero or one elements and is therefore already sorted.

$\text{SORT}(a, b)$ is transformed to:

```
if $a \geq b$ then $\text{SORT}(a, b)$
else $\text{SORT}(a, b)$ fi
```

Add assertions:

```
if $a \geq b$ then \{a $\geq$ b\}; $\text{SORT}(a, b)$
else \{a $<$ b\}; $\text{SORT}(a, b)$ fi
```
Elaboration

For the sort function, the simplest case is when \( a \geq b \). In this case, the array has zero or one elements and is therefore already sorted.

\[
\text{SORT}(a, b) \text{ is transformed to:}
\]

\[
\begin{align*}
\text{if } a \geq b \text{ then } & \text{ SORT}(a, b) \\
\text{else } & \text{ SORT}(a, b) \text{ fi}
\end{align*}
\]

Add assertions:

\[
\begin{align*}
\text{if } a \geq b \text{ then } & \{ a \geq b \}; \text{ SORT}(a, b) \\
\text{else } & \{ a < b \}; \text{ SORT}(a, b) \text{ fi}
\end{align*}
\]

Use the assertions:

\[
\begin{align*}
\text{if } a \geq b \text{ then } & \{ a \geq b \}; \text{ skip} \\
\text{else } & \{ a < b \}; \text{ SORT}(a, b) \text{ fi}
\end{align*}
\]
Divide and Conquer

Use the informal implementation ideas to direct the selection of transformations.
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For the factorial program the “idea” is to use the definition of the factorial function when $n > 0$:

$$n! = n.(n - 1)!$$
Divide and Conquer

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For the factorial program the "idea" is to use the definition of the factorial function when \( n > 0 \):

\[
 n! = n.(n - 1)!
\]

Applying this idea to the elaborated specification we get:

\[
\text{if } n = 0 \text{ then } r := 1 \\
\text{else } r := (n - 1)!; \ r := n.r \ 
\text{fi}
\]
Divide and Conquer

Use the informal implementation ideas to direct the selection of transformations.

For the factorial program the “idea” is to use the definition of the factorial function when \( n > 0 \):

\[
\frac{n!}{(n-1)!} = n.
\]

Applying this idea to the elaborated specification we get:

\[
\text{if } n = 0 \text{ then } r := 1 \\
\text{else } r := (n-1)!, \ r := n.r \ \text{fi}
\]

The statement \( r := (n-1)! \) is expanded into three statements:

\[
\begin{align*}
&n := n - 1; \\
&r := n!; \\
&n := n + 1
\end{align*}
\]

Note that this contains a copy of the original specification.
Divide and Conquer

For Quicksort the implementation idea has two stages:
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1. Partition the array around a pivot element: so that all the elements less than the pivot go on one side and the larger elements go on the other side
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2. Sort the two sub-arrays sorted using copies of the original specification statement.
For Quicksort the implementation idea has two stages:

1. Partition the array around a pivot element: so that all the elements less than the pivot go on one side and the larger elements go on the other side

2. Sort the two sub-arrays sorted using copies of the original specification statement.

This leads to the program:

\[
\text{if } a < b \text{ then Partition}(a, b, m); \\
\text{SORT}(a, m - 1); \\
\text{SORT}(m + 1, b) \text{ fi}
\]

In this case, the elaborated specification contains two copies of the original specification
Recursion Introduction

Apply the Recursive Implementation Theorem to produce a recursive procedure.

The Recursive Implementation Theorem can be applied when:

1. The elaborated specification is a refinement of the original specification; and

2. There exists a variant function which is reduced before each copy of the original specification

If both these conditions are satisfied, then the elaborated specification can be transformed into a recursive procedure, with each copy of the original specification replaced by a recursive call.
Recursion Introduction

For the factorial program, the elaborated specification is:

\[
\text{if } n = 0 \text{ then } r := 1 \\
\quad \text{else } n := n - 1; \ r := n!; \ n := n + 1; \ r := n \cdot r \ \text{fi}
\]

This is equivalent to, and contains one copy of, the original specification \( r := n! \).

Also, the value of \( n \) is a non-negative integer and is reduced before the copy of the specification.

If an elaborated specification is equivalent to the original specification and there is an expression whose value is reduced before each copy of the specification, then it can be refined to a recursive procedure with the internal copies of the specification replaced by recursive calls.
Recursion Introduction

For the factorial program we get this recursive procedure:

\[
\text{proc } \text{fact } \equiv \\
\quad \text{if } n = 0 \text{ then } r := 1 \\
\quad \quad \text{else } n := n - 1; \text{ fact; } n := n + 1; \ r := n.r \ \text{fi}
\]

The value of the variant function must be “smaller” in terms of a well-founded partial order on some set of values. Typically this will be a non-negative integer, but other possibilities include a subset order and a lexical order on a list of integers.
Recursion Introduction

More formally:

If $\preceq$ is a well-founded partial order on some set $\Gamma$ and $t$ is a term giving values in $\Gamma$ and $t_0$ is a variable which does not occur in $S$ or $S'$ then if

$$\{P \land t \preceq t_0\}; \ S \leq S'[\{P \land t < t_0\}; \ S/X]$$

then $\{P\}; \ S \leq \text{proc } X \equiv S' \text{ end}$

Here, $S$ is the original specification which is elaborated to $S'[S/X]$. $P$ is any required precondition: if no precondition is needed, then let $P$ be true.

The variant function is $t$. If the value of $t$ is initially no larger than $t_0$, the before each copy of the specification we know that $t$ is strictly less than $t_0$. 
Recursion Removal

Suppose we have a recursive procedure whose body is a regular action system in the following form:

```
proc F(x) ≡
    actions A_1:
        A_1 ≡ S_1.
        ...A_i ≡ S_i.
        ...B_j ≡ S_{j0}; F(g_{j1}(x)); S_{j1}; F(g_{j2}(x));
        ...; F(g_{jn_j}(x)); S_{jn_j}.
    ... endactions.
```

where \( S_{j1}, \ldots, S_{jn_j} \) preserve the value of \( x \) and no \( S \) contains a call to \( F \) and the statements \( S_{j0}, S_{j1}, \ldots, S_{jn_j - 1} \) contain no action calls.

Note: Any action system can be converted into this form using the destructuring and restructuring transformations.
Stack $L$ records “postponed” operations

A postponed call $F(e)$ is recorded by pushing $\langle 0, e \rangle$ onto $L$

A postponed execution of $S_{jk}$ is recorded by pushing the value $\langle \langle j, k \rangle, x \rangle$ onto $L$.

When then procedure body would normally terminate (via call $Z$) we call a new action $\hat{F}$ which pops the top item off $L$ and carries out the postponed operation.

If we call $\hat{F}$ with the stack empty then all postponed operations have been completed and the procedure terminates by calling $Z$. 
Recursion Removal

\[
\text{proc } F'(x) \equiv \\
\text{var } \langle L := \langle \rangle, m := 0 \rangle: \\
\text{actions } A_1: \\
A_1 \equiv S_1[\text{call } \hat{F}/\text{call } Z]. \\
\ldots A_i \equiv S_i[\text{call } \hat{F}/\text{call } Z]. \\
\ldots B_j \equiv S_{j0}; \\
L := \langle\langle j, 1 \rangle, x \rangle, \langle 0, g_{j2}(x) \rangle, \\
\ldots, \langle 0, g_{jn_j}(x) \rangle, \langle j, n_j \rangle, x \rangle \rangle \ + \ L; \\
x := g_{j1}; \\
\text{call } A_j. \\
\ldots \hat{F} \equiv \text{if } L = \langle \rangle \\
\text{then call } Z \\
\text{else } \langle m, x \rangle \overset{\text{pop}}{\leftarrow} L; \\
\text{if } m = 0 \rightarrow \text{call } A_1 \\
\square \ldots \square m = \langle j, k \rangle \rightarrow S_{jk}[\text{call } \hat{F}/\text{call } Z]; \text{ call } \hat{F} \\
\ldots \text{ fi fi. endactions end.}
Consider the special case of a parameterless, linear recursion:

\[
\text{proc } F \equiv \\
\begin{align*}
\text{actions } & A_1: \\
A_1 & \equiv S_1. \\
\dotsc A_i & \equiv S_i, \dotsc \\
B_1 & \equiv S_0; \ F; \ S_{11}. \ 	ext{endactions.}
\end{align*}
\]

After applying the general recursion removal theorem, the only value pushed into the stack is \(\langle\langle 1, 1 \rangle\rangle\). So the stack can be replaced by an integer which records how many values are on the stack,
Linear Recursion Removal

The iterative program is:

\[
\text{proc } F' \equiv \\
\text{var } \langle L := 0 \rangle : \\
\text{actions } A_1 : \\
A_1 \equiv S_1[\text{call } \hat{F}/\text{call } Z]. \\
\ldots A_i \equiv S_i[\text{call } \hat{F}/\text{call } Z]. \\
B_1 \equiv S_{j0}; L := L + 1; \text{ call } A_1. \\
\hat{F} \equiv \text{if } L = 0 \\
\text{ then call } Z \\
\text{ else } L := L - 1; \\
\quad S_{11}[\text{call } \hat{F}/\text{call } Z]; \text{ call } \hat{F} \text{ fi. endactions end.}
\]
Linear Recursion Removal

For example:

```
proc F ≡
    if B then S₁ else S₂; F; S₃ fi.
```

is equivalent to the iterative program:

```
proc F' ≡
    var ⟨L := 0⟩:
    actions A₁:
        A₁ ≡ if B then S₁; call F else call B₁ fi.
        B₁ ≡ S₂; L := L + 1; call A₁.
        ̂F ≡ if L = 0 then call Z
            else L := L − 1;
                S₃; call ̂F fi. endactions end.
```
Linear Recursion Removal

Remove the recursion in \( \hat{F} \), unfold into \( A_1 \), unfold \( B_1 \) into \( A_1 \) and remove the recursion to give:

\[
\text{proc } F' \equiv \\
\text{var } \langle L := 0 \rangle: \\
\quad \text{while } \neg B \text{ do } S_2; \ L := L + 1 \text{ od; }
\quad S_1; \\
\quad \text{while } L \neq 0 \text{ do } L := L - 1; \ S_3 \text{ od.}
\]

This restructuring is carried out automatically by FermaT’s Collapse_Action_System transformation.
Recursion Removal Example

For the factorial program we derived this recursive procedure:

\[
\text{proc } \text{fact } \equiv \\
\quad \text{if } n = 0 \text{ then } r := 1 \\
\quad \text{else } n := n - 1; \text{ fact;} \quad n := n + 1; \quad r := n.r \text{ fi}
\]

This transforms to the equivalent iterative procedure:

\[
\text{proc } F' \equiv \\
\quad \text{var } \langle L := 0 \rangle : \\
\quad \text{while } n \neq 0 \text{ do } n := n - 1; \quad L := L + 1 \text{ od;} \\
\quad r := 1; \\
\quad \text{while } L \neq 0 \text{ do } L := L - 1; \quad n := n + 1; \quad r := n.r \text{ od.}
\]

The first loop just copies \( n \) to \( L \) and sets \( n \) to zero.

The second loop iterates \( n \) from 1 to \( L \) (which was the initial value of \( n \)).
Recursion Removal Example

proc \( F' \equiv \)
\[
\text{var } \langle L := n \rangle :
\]
\[
n := 0;
\]
\[
r := 1;
\]
\[
\textbf{while } L \neq 0 \textbf{ do } L := L - 1; \ n := n + 1; \ r := n.r \ \textbf{od}.
\]

If we add a new variable \( n_0 \) to record the initial value of \( n \) then \( L \) is not needed since the test \( L \neq 0 \) can be replaced by the equivalent test \( n \neq n_0 \):

proc \( F' \equiv \)
\[
\text{var } \langle n_0 := n \rangle :
\]
\[
n := 0;
\]
\[
r := 1;
\]
\[
\textbf{while } n \neq n_0 \textbf{ do } n := n + 1; \ r := n.r \ \textbf{od}.
\]
Recursion Removal Example

The result can be written as a for loop:

\[
\text{proc } F' \equiv \\
\quad r := 1; \\
\quad \text{for } i := 1 \text{ to } n \text{ do } r := i \cdot r \text{ od.}
\]
Selection Sort

Define the predicate $\text{Sorted}(A, i, j)$ to be true precisely when the array segment $A[i..j]$ is sorted:

$$\text{Sorted}(A, i, j) =_{DF} \forall k. i \leq k < j \Rightarrow A[k] \leq A[k + 1]$$

Define the predicate $\text{Perm}(A, A')$ to mean that the elements in array $A$ form a permutation of the elements in array $A'$.

The formal specification for a sorting program can now be written as follows:

$$\text{SORT}(A, i, j) =_{DF} \begin{array}{c} A[i..j] := A'[i..j].(\text{Sorted}(A', i, j) \land \text{Perm}(A[i..j], A'[i..j])) \end{array}$$
If $i \geq j$ then the array has at most one element and is therefore already sorted. So in this case:

$$\text{SORT}(A, i, j) \approx \text{skip}$$

So we can elaborate the specification to:

```
if $i < j$ then SORT($A, i, j$) fi
```
Selection Sort: Informal Idea

The informal idea behind selection sorting is: “find the smallest element in the array, and move it to the front”.

Inserting any permutation of $A[i..j]$ before a copy of $\text{SORT}(A, i, j)$ has no effect, so $\text{SORT}(A, i, j)$ is equivalent to:

\[
\text{if } i < j \\
\text{ then } \text{var} \langle s := 0 \rangle : \\
\quad s := s'.(i \leq s' \leq j \land \forall k. i \leq k \leq j \Rightarrow A[s'] \leq A[k]); \\
\quad \langle A[i], A[s] \rangle := \langle A[s], A[i] \rangle \text{ end}; \\
\text{SORT}(A, i, j) \text{ fi}
\]

With this addition to the program, we have the assertion: $\forall k. i \leq k \leq j \Rightarrow A[i] \leq A[k]$ just before the copy of $\text{SORT}$.

So $\text{SORT}(A, i, j)$ can be refined as $\text{SORT}(A, i + 1, j)$
Selection Sort: Recursion Introduction

The expression $j - i$ is positive, and is reduced before the copy of SORT, so we can apply Recursion_Introduction to get this recursive program:

```
proc sort ≡
    if $i < j$
        then var $\langle s := 0 \rangle$
            $s := s'.(i \leq s' \leq j \land \forall k. i \leq k \leq j \Rightarrow A[s'] \leq A[k]);$
            $\langle A[i], A[s] \rangle := \langle A[s], A[i] \rangle$ end;
            $i := i + 1;$
            sort fi.
```
Selection Sort: Recursion Introduction

This is equivalent to the while loop:

\[
\begin{align*}
\text{proc sort} & \equiv \\
\text{while } i < j \text{ do} & \\
\quad \var \langle s := 0 \rangle : & \\
\quad s := s'.(i \leq s' \leq j \land \forall k. i \leq k \leq j \Rightarrow A[s'] \leq A[k]); & \\
\quad \langle A[i], A[s] \rangle := \langle A[s], A[i] \rangle \text{ end;} & \\
\quad i := i + 1 \text{ od.} &
\end{align*}
\]
Selection Sort: Refinement

To implement the inner specification statement, first take out a trivial case:

if $i = j$ then the specification can be implemented as the assignment $s := i$:

$$\begin{align*}
\text{if } i &= j \\
\quad \text{then } s &= i \\
\quad \text{else } s &= s'.(i \leq s' \leq j \land \forall k. i \leq k \leq j \Rightarrow A[s'] \leq A[k]) \quad \text{fi}
\end{align*}$$

Our informal idea for implementing the specification is to first set $s$ to the index of the smallest element in $A[i..j-1]$ and then compare $A[s]$ against $A[j]$. 
Selection Sort: Refinement

This produces the elaborated specification:

\[
\begin{align*}
\text{if } i &= j \\
\text{then } s &:= i \\
\text{else } j &:= j - 1; \\
& \quad s := s'.(i \leq s' \leq j \land \forall k. i \leq k \leq j \Rightarrow A[s'] \leq A[k]); \\
& \quad j := j + 1; \\
& \quad \text{if } A[j] < A[s] \text{ then } s := j \fi \fi
\end{align*}
\]

The variable \( j \) is our variant function, so we can apply Recursion_Introduction on the copy of the specification:

\[
\text{proc search } \equiv
\begin{align*}
\text{if } i &= j \\
\text{then } s &:= i \\
\text{else } j &:= j - 1; \text{ search; } j := j + 1; \\
& \quad \text{if } A[j] < A[s] \text{ then } s := j \fi \fi.
\end{align*}
\]
Selection Sort: Refinement

Apply Recursion_Removal to get:

\[ \text{proc search } \equiv \]
\[ \text{var } \langle L := 0 \rangle : \]
\[ \text{while } i \neq j \text{ do} \]
\[ j := j - 1; \quad L := L + 1 \text{ od}; \]
\[ s := i; \]
\[ \text{while } L \neq 0 \text{ do} \]
\[ L := L - 1; \quad j := j + 1; \]
\[ \text{if } A[j] < A[s] \text{ then } s := j \text{ fi od end.} \]
Selection Sort: Refinement

As above, the variable $L$ is incremented whenever $j$ is decremented, and vice-versa. So if $j_0$ is the original value of $j$ then $L = j_0 - j$. The first loop assigns $j := i$:

\[
\text{proc search } \equiv \\
\text{var } \langle j_0 := j \rangle : \\
\quad j := i; \\
\quad s := i; \\
\quad \text{while } j < j_0 \text{ do} \\
\quad \quad j := j + 1; \\
\quad \quad \text{if } A[j] < A[s] \text{ then } s := j \text{ fi od end.}
\]
Selection Sort: Refinement

Putting this into the sorting program, instead of the specification we get the completed program:

\[
\textproc{sort} \equiv \\
\quad \text{while } i < j \text{ do} \\
\quad \quad \text{var} \langle s := i, j_0 := j \rangle : \\
\quad \quad \quad j := i; \\
\quad \quad \quad \text{while } j < j_0 \text{ do} \\
\quad \quad \quad \quad j := j + 1; \\
\quad \quad \quad \quad \text{if } A[j] < A[s] \text{ then } s := j \text{ fi od; } \\
\quad \quad \quad \langle A[i], A[s] \rangle := \langle A[s], A[i] \rangle \text{ end;} \\
\quad \quad i := i + 1 \text{ od.}
\]
String Comparison

Given two character strings $a$ and $b$, it required to determine whether they are equal “apart from blanks” (the space character being regarded as non-significant).

We represent the strings as arrays of characters, with the special symbol end denoting the end of the string.
Given two character strings $a$ and $b$, it is required to determine whether they are equal “apart from blanks” (the space character being regarded as non-significant).

We represent the strings as arrays of characters, with the special symbol `end` denoting the end of the string.

Define the function $\text{strip}(s, i)$ to return the sequence of all non-space characters in $s$ from the $i$th character to the end of the string:

\[
\text{strip}(s, i) = \begin{cases} 
\langle \rangle & \text{if } s[i] = \text{end} \\
\text{strip}(s, i + 1) & \text{if } s[i] = \text{space} \\
\langle s[i] \rangle + \text{strip}(s, i + 1) & \text{otherwise}
\end{cases}
\]
Formal Specification

With this definition of strip our formal specification is:

\[
\text{COMP} \overset{\text{DF}}{=} \text{if } \text{strip}(a, 1) = \text{strip}(b, 1) \text{ then } R := 1 \text{ else } R := 0 \text{ fi}
\]
Informal Ideas

Our informal idea is to step through both arrays a character at a time until we reach the end, or find a significant difference. This suggests generalising the specification to compare the strings from a given index onwards:

\[
\text{COMP}(i, j) =_{DF} \begin{cases} 
\text{if } \text{strip}(a, i) = \text{strip}(b, j) \text{ then } R := 1 \\
\text{else } R := 0 
\end{cases}
\]
Elaborated Specification

The obvious special cases to consider are the values of $a[i]$ and $b[j]$. 
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First we consider the case where $a[i] = \text{space}$:

```plaintext
if $a[i] = \text{space}$ then COMP($i, j$)
    else COMP($i, j$) fi
```
The obvious special cases to consider are the values of $a[i]$ and $b[j]$.

First we consider the case where $a[i] = \text{space}$:

```
if $a[i] = \text{space}$ then COMP($i, j$)
else COMP($i, j$) fi
```

By definition, if $a[i] = \text{space}$ then $\text{strip}(a, i) = \text{strip}(a, i + 1)$ so $\text{COMP}(i, j) \approx \text{COMP}(i + 1, j)$.

We have:

```
if $a[i] = \text{space}$ then COMP($i + 1, j$)
else COMP($i, j$) fi
Recursive Implementation

Let \( i' \) be the first index such that \( a[i] = \text{end} \).

Then the variant function \( i' - i \) is reduced before the first copy of the specification, but (obviously) not before the second copy.

We can still apply Recursive Implementation, provided we only apply it to the \textit{first} copy of the specification:

\[
\text{proc comp } \equiv \\
\quad \text{if } a[i] = \text{space} \text{ then } i := i + 1; \text{ comp} \\
\quad \text{else } \text{COMP}(i,j) \text{ fi}
\]
Recursion Removal

This simple tail-recursion is transformed to a **while** loop:

```plaintext
while a[i] = space do i := i + 1 od;
COMP(i, j)
```
Recursion Removal

This simple tail-recursion is transformed to a while loop:

while \( a[i] = \text{space} \) do \( i := i + 1 \) od;
COMP\((i, j)\)

A similar argument for \( b[j] \) produces:

while \( a[i] = \text{space} \) do \( i := i + 1 \) od;
while \( b[j] = \text{space} \) do \( j := j + 1 \) od;
COMP\((i, j)\)
Further Refinement

Consideration of the cases where \( a[i] = \text{end} \) and/or \( b[j] = \text{end} \) gives:

\[
\begin{align*}
\text{while } a[i] = \text{space} & \text{ do } i := i + 1 \text{ od;} \\
\text{while } b[j] = \text{space} & \text{ do } j := j + 1 \text{ od;} \\
\text{if } a[i] = \text{end} & \land b[j] = \text{end} \text{ then } R := 1 \\
\text{elsif } a[i] \neq a[j] & \text{ then } R := 0 \\
\text{else } & i := i + 1; \ j := j + 1; \ \text{COMP}(i, j) \text{ fi}
\end{align*}
\]
Final Program

Apply Recursive_Implementation and Recursion_Removal to get the final iterative program:

\[
\begin{align*}
d \text{o while} & \ a[i] = \text{space} \ \text{d}o \ \ i := i + 1 \ \text{od}; \\
& \text{while} \ b[j] = \text{space} \ \text{d}o \ \ j := j + 1 \ \text{od}; \\
& \text{if} \ a[i] = \text{end} \land b[j] = \text{end} \ \text{then} \ R := 1; \ \text{exit}(1) \\
& \text{elsif} \ a[i] \neq a[j] \ \text{then} \ R := 0; \ \text{exit}(1) \ \text{fi;} \\
& i := i + 1; \ j := j + 1 \ \text{od}
\end{align*}
\]
The Greatest Common Divisor (GCD) of two numbers is the largest number which divides both of the numbers with no remainder.

A specification for a program which computes the GCD is the following:

\[ r := \text{GCD}(x, y) \]

where:

\[ \text{GCD}(x, y) = \max \{ n \in \mathbb{N} | x \mod n = 0 \land y \mod n = 0 \} \]

(Note that this is undefined when both \( x \) and \( y \) are zero).
Implementation Ideas

It is easy to prove the following facts about GCD:
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Implementation Ideas

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It is easy to prove the following facts about GCD:

1. $\text{GCD}(0, y) = y$
2. $\text{GCD}(x, 0) = x$
3. $\text{GCD}(x, y) = \text{GCD}(y, x)$
4. $\text{GCD}(x, y) = \text{GCD}(-x, y) = \text{GCD}(x, -y)$
Implementation Ideas

It is easy to prove the following facts about GCD:

1. \( \text{GCD}(0, y) = y \)
2. \( \text{GCD}(x, 0) = x \)
3. \( \text{GCD}(x, y) = \text{GCD}(y, x) \)
4. \( \text{GCD}(x, y) = \text{GCD}(-x, y) = \text{GCD}(x, -y) \)
5. \( \text{GCD}(x, y) = \text{GCD}(x - y, y) = \text{GCD}(x, y - x) \)
Split on the conditions \( x = 0 \) and \( y = 0 \), using Fact (1) and Fact (2) respectively to show \( r := \text{GCD}(x, y) \) is refined by:

\[
\text{if } x = 0 \text{ then } r := y \\
\text{elsif } y = 0 \text{ then } r := x \\
\text{else } r := \text{GCD}(x, y) \text{ fi}
\]
Split on the conditions $x = 0$ and $y = 0$, using Fact (1) and Fact (2) respectively to show $r := \text{GCD}(x, y)$ is refined by:

\[
\text{if } x = 0 \text{ then } r := y \\
\text{elsif } y = 0 \text{ then } r := x \\
\text{else } r := \text{GCD}(x, y) \text{ fi}
\]

Fact (3) does not appear to make much progress.
Split on the conditions $x = 0$ and $y = 0$, using Fact (1) and Fact (2) respectively to show $r := \text{GCD}(x, y)$ is refined by:

```plaintext
if $x = 0$ then $r := y$
elsif $y = 0$ then $r := x$
else $r := \text{GCD}(x, y)$ fi
```

Fact (3) does not appear to make much progress.

If we restrict attention to non-negative integers, then Fact (4) does not apply. So we are left with Fact (5).
Split on the conditions \( x = 0 \) and \( y = 0 \), using Fact (1) and Fact (2) respectively to show \( r := \text{GCD}(x, y) \) is refined by:

\[
\begin{align*}
\text{if } x = 0 & \text{ then } r := y \\
\text{elsif } y = 0 & \text{ then } r := x \\
\quad \text{else } r := \text{GCD}(x, y) & \text{ fi}
\end{align*}
\]

Fact (3) does not appear to make much progress. If we restrict attention to non-negative integers, then Fact (4) does not apply. So we are left with Fact (5).

We can only transform \( r := \text{GCD}(x, y) \) to \( r := \text{GCD}(x - y, y) \) under the condition \( x \geq y \).
Split on the conditions $x = 0$ and $y = 0$, using Fact (1) and Fact (2) respectively to show $r := \text{GCD}(x, y)$ is refined by:

```
if $x = 0$ then $r := y$
elsif $y = 0$ then $r := x$
else $r := \text{GCD}(x, y)\fi$
```

Fact (3) does not appear to make much progress.

If we restrict attention to non-negative integers, then Fact (4) does not apply. So we are left with Fact (5).

We can only transform $r := \text{GCD}(x, y)$ to $r := \text{GCD}(x - y, y)$ under the condition $x \geq y$.

Similarly, we can only transform $r := \text{GCD}(x, y)$ to $r := \text{GCD}(x, y - x)$ under the condition $y \geq x$. 
Elaborated Specification

We have the following elaboration of the specification:

\[
\text{if } x = 0 \\
\quad \text{then } r := y \\
\text{elsif } y = 0 \\
\quad \text{then } r := x \\
\text{elsif } x \geq y \text{ then } r := \text{GCD}(x - y, y) \\
\quad \text{else } r := \text{GCD}(x, y - x) \text{ fi}
\]

The variant function \( x + y \) is reduced before each copy of the specification.
Recursion Introduction

Applying the recursion introduction gives:

```
proc gcd(x, y) ≡
  if x = 0
    then r := y
  elsif y = 0
    then r := x
  elsif x ⩾ y then r := gcd(x − y, y)
    else r := gcd(x, y − x) fi end
```

Recursion Removal gives:

```
proc gcd(x, y) ≡
  while x ≠ 0 ∧ y ≠ 0 do
    if x ⩾ y then x := x − y
      else y := y − x fi od;
  if x = 0 then r := y else r := x fi end
```
Optimisation

This algorithm, although correct, is very inefficient when $x$ and $y$ are very different in size. For example, if $x = 1$ and $y = 2^{31}$ then the program will take $2^{31} - 1$ steps.

One solution is to look for other properties of GCD to use: this involves throwing away all our work so far.

Unfortunately, this is the only option offered by the “Invariant Based Programming” approach.
Optimisation

With the transformational programming approach, we have another option: transform the program in order to improve its efficiency.

Apply Entire Loop Unrolling to the program at the point just after the assignment $x := x - y$ with the condition $x \geq y$:

```plaintext
proc gcd(x, y) ≡
  while x ≠ 0 ∧ y ≠ 0 do
    if x ≥ y then x := x − y;
      while x ≥ y do
        if x ≥ y then x := x − y fi od
      else y := y − x fi od;
  r := x end
```
Optimisation

This simplifies to:

\[
\text{proc } \gcd(x, y) \equiv \\
\quad \text{while } x \neq 0 \land y \neq 0 \text{ do} \\
\qquad \text{if } x \geq y \text{ then while } x \geq y \text{ do } x := x - y \text{ od} \\
\qquad \quad \text{else } y := y - x \text{ fi od;} \\
\qquad \text{if } x = 0 \text{ then } r := y \text{ else } r := x \text{ fi end}
\]

Consider the inner \texttt{while} loop. This repeatedly subtracts \( y \) from \( x \).

If the loop executes \( q \) times, then \( x = x_0 - q.y \).

In other words:

\[
\text{while } x \geq y \text{ do } x := x - y \text{ od } \approx x := x \mod y
\]
Similarly, entire loop unrolling can be applied after the assignment $y := y - x$ and the same optimisation applied to give:

```plaintext
proc gcd(x, y) ≡
    while $x \neq 0 \land y \neq 0$ do
        if $x \geq y$ then $x := x \mod y$
            else $y := y \mod x$ fi od;
    if $x = 0$ then $r := y$ else $r := x$ fi end
```
Alternate Program Derivation

With different informal ideas, the same derivation process can derive a different algorithm.

For example, suppose the target machine does not have an efficient integer division instruction, but does have a binary shift.

We make use of the following additional facts about GCD:

1. $\text{GCD}(x, y) = 2 \cdot \text{GCD}(x/2, y/2)$ when $x$ and $y$ are both even;
2. $\text{GCD}(x, y) = \text{GCD}(x/2, y)$ when $x$ is even and $y$ is odd;
3. $\text{GCD}(x, y) = \text{GCD}(x, y/2)$ when $x$ is odd and $y$ is even;
4. $\text{GCD}(x, y) = \text{GCD}((x - y)/2, y)$ when $x$ and $y$ are odd and $x \geq y$;
5. $\text{GCD}(x, y) = \text{GCD}(x, (y - x)/2)$ when $x$ and $y$ are odd and $y \geq x$. 
Applying Fact (I) above produces:

\[
\begin{align*}
\text{if } x = 0 & \text{ then } r := y \\
\text{elsif } y = 0 & \text{ then } r := x \\
\text{elsif even?(} x \text{) } \land \text{ even?(} y \text{)} & \\
\quad & \text{ then } r := 2 \cdot \text{GCD}(x/2, y/2) \\
\quad & \text{ else } r := \text{GCD}(x, y) \quad \text{fi}
\end{align*}
\]
Recursive Implementation

Applying the recursive implementation theorem plus recursion removal to the first occurrence only of GCD produces:

\[
\text{if } x = 0 \\
\quad \text{then } r := y \\
\text{elsif } y = 0 \\
\quad \text{then } r := x \\
\quad \text{else var } \langle L := 0 \rangle : \\
\quad \quad \text{while even?}(x) \land \text{even?}(y) \text{ do} \\
\quad \quad \quad L := L + 1; \\
\quad \quad \quad x := x/2; \; y := y/2 \text{ od;} \\
\quad \quad r := \text{GCD}(x, y); \\
\quad \quad r := 2^L \cdot r \text{ end fi}
\]
Applying Fact (Ⅰ) above, followed by recursion introduction and recursion removal produces the following result:

if $x = 0$ then $r := y$
elsif $y = 0$ then $r := x$
else var $\langle L := 0 \rangle$:
    while even?(x) $\wedge$ even?(y) do
        $L := L + 1$
        $x := x/2$; $y := y/2$ od;
    while even?(x) do $x := x/2$ od;
\{ $x \neq 0 \wedge y \neq 0 \wedge \neg$even?(x)\};
$r := \text{GCD}(x, y)$;
$r := 2^L \cdot r$ end fi
Define:

\[ \text{GCD}_x(x, y) =_{DF} \{ y \neq 0 \land \neg \text{even?}(x) \}; \ r := \text{GCD}(x, y) \]

By Fact (3) we show that \( \text{GCD}_x(x, y) \) is equivalent to:

\begin{verbatim}
while even?(y) do y := y/2 od;
GCDx(x, y)
\end{verbatim}
Recursive Implementation

Now apply Fact (4), and also Fact (3) from the first set of facts, to ensure that $x$ is odd in every occurrence of GCDx:

```plaintext
while even?(y) do y := y/2 od;
if x = y then r := x
elsif x > y then GCDx(y, (x - y)/2)
   else GCDx(x, (y - x)/2) fi
```
Recursive Implementation

Apply recursion introduction and recursion removal to derive this implementation of \( \text{GCD}^x(x, y) \):

\[
\text{do while even?}(y) \text{ do } y := y/2 \text{ od;}
\]

\[
\text{ if } x = y \text{ then } r := x; \text{ exit fi;}
\]

\[
\text{ if } x > y \text{ then}
\]

\[
\text{ then } \langle x, y \rangle := \langle y, x - y \rangle
\]

\[
\text{ else } y := y - x \text{ fi;}
\]

\[
y := y/2 \text{ od}
\]
Recursive Implementation

The final program is therefore:

```plaintext
if x = 0 then r := y
elsif y = 0 then r := x
else var ⟨L := 0⟩:
  while even?(x) ∧ even?(y) do
    L := L + 1;
    x := x/2; y := y/2 od;
  while even?(x) do x := x/2 od;
  do while even?(y) do y := y/2 od;
  if x = y then r := x; exit fi;
  if x > y
    then ⟨x, y⟩ := ⟨y, x − y⟩
    else y := y − x fi;
  y := y/2 od
r := 2^L.r end fi
```