ON THE DIAMETER OF THE ATTRACTOR OF AN IFS

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Abstract

We investigate methods for the evaluation of the diameter of the attractor of an IFS. We propose an upper bound for the diameter in n-dimensional space. In the case of an affine IFS, we indicate how this upper bound can be calculated.

Key Words: attractors, diameter, fractals.
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1 Introduction

Apart from its theoretical interest, the evaluation of the diameter of the attractor of an iterated function system (IFS) is useful in many algorithms. Dubuc-Elqortobi [3] have considered adaptive methods to reduce the computations in the deterministic algorithm [1] and Hepting-Prusinkiewicz-Saupe [5] have proposed an algorithm that renders the geometry of the space which contains an attractor characterizing each point according to its distance to the attractor. In these methods it is assumed that the diameter of the attractor or at least an upper bound is known. However, no clue was provided on how to determine this diameter although it is clear that computer graphic tests can lead to good approximations. In this paper, we propose a class of upper bounds for the diameter of the attractor of an IFS in a complete metric space. The existence of a smallest such upper bound is proved in n-dimensional space. If the IFS is affine, we suggest a method which enables us to compute this smallest upper bound. Finally, we show that in a particular case the diameter of the attractor is equal to the diameter of the convex hull of the set of fixed points determined by the IFS.

2 Approximation of the diameter of an IFS

We recall first some basic results of IFS theory. Let (X, d) be a complete metric space. The class of all non-empty closed bounded subsets of X is a complete metric space for the Hausdorff metric h [7]. Given N contractions $f_1, f_2, \ldots, f_N : X \to X$, the finite set $F = \{f_1, f_2, \ldots, f_N\}$ is called an iterated function system or IFS. Let $L(f_i)$ denote the Lipschitz constant of the contraction f_i, $i = 1, 2, \ldots, N$. The following theorem can be found in Hutchinson [6].

Theorem 1 *Let $F = \{f_1, f_2, \ldots, f_N\}$ be an IFS of the complete metric space (X, d). Then there exists a unique non-empty compact set $\mathcal{A}$ called attractor of the IFS such that $\mathcal{A} = F(\mathcal{A}) = \bigcup_{i=1}^{N} f_i(\mathcal{A})$.*

Lemma 2 *Let $F = \{f_1, f_2, \ldots, f_N\}$ be an IFS of the complete metric space (X, d) with attractor $\mathcal{A}$. Let E be a non-empty closed bounded set such that $f_i(E) \subset E$ for each i. Then $\mathcal{A} \subset E$.*

A proof is given in [4].

Proposition 3 *Let $F = \{f_1, f_2, \ldots, f_N\}$ be an IFS of the complete metric space (X, d) with attractor $\mathcal{A}$. Then for any $x \in X$, $\mathcal{A} \subset B(x, r_x)$, where $r_x = \max_{1 \le i \le N} \frac{d(x, f_i(x))}{1 - L(f_i)}$.*

Proof. If x_0 is in the closed ball $B(x, r_x)$, then for every $i \in \{1, 2, \ldots, N\}$, $f_i(x_0) \in B(x, r_x)$. This is a consequence of the following inequalities

$$\begin{aligned} d(f_i(x_0), x) &\le d(f_i(x_0), f_i(x)) + d(f_i(x), x) \\ &\le L(f_i) r_x + d(f_i(x), x) \\ &\le r_x. \end{aligned}$$

Thus, $f_i\{B(x, r_x)\} \subset B(x, r_x)$ which implies that $\mathcal{A} \subset B(x, r_x)$ by Lemma 2. □

Proposition 4 *Let $F = \{f_1, f_2, \ldots, f_N\}$ be an IFS of $(\mathbb{R}^n, d)$. We define $\Phi : \mathbb{R}^n \to \mathbb{R}$ by $\Phi(x) = r_x$. Then Φ is continuous and has a global minimum.*

Proof. Φ is clearly continuous. Let Ω_1 be the fixed point of f_1. For each x not contained in $B(\Omega_1, r_{\Omega_1})$

$$\begin{aligned} \Phi(\Omega_1) &< d(x, \Omega_1) \qquad (1) \\ &\le \Phi(x), \qquad (2) \end{aligned}$$

where the inequality (2) is justified by the fact that $\Omega_1 \in \mathcal{A} \subset B(x, r_x)$. Since Φ is continuous on the compact $B(\Omega_1, r_{\Omega_1})$, it reaches on this set a lower bound which is also the global minimum due to (2). □

We deduce from Proposition 4 that from all balls $B(x, r_x)$ that contain the attractor, there is at least one with smallest diameter. This diameter is our best upper bound for the diameter of the attractor.

From now on we shall consider only affine IFSs of n-dimensional space, i.e., IFSs such that the contractions are affine transformations of $\mathbb{R}^n$.

What can be said about the nature of the set of points at which the minimum of the function Φ is reached ?

Proposition 5 *The set of solutions of the minimization problem* $\inf_{u \in \mathbb{R}^n} \Phi(u)$ *is a closed convex set and consists of a unique point if the function* Φ *is strictly convex over* $\mathbb{R}^n$.

Proof. We note first that Φ is convex. This results from the convexity of each function $x \to \frac{d(x, f_i(x))}{1 - L(f_i)}$ which is clear since

$$\begin{aligned} d(x, f_i(x)) &= \|x - f_i(x)\| && (3) \\ &= \|A_i x + B_i\|, && (4) \end{aligned}$$

where A_i is a linear map and B_i is a vector. Let S be the set of solutions of the minimization problem $\inf_{u \in \mathbb{R}^n} \Phi(u)$. We know from Proposition 4 that $S \neq \emptyset$. Moreover, S is closed since Φ is continuous. If both x and y attain $m = \inf_{u \in \mathbb{R}^n} \Phi(u)$, then

$$m \leq \Phi(\lambda x + (1 - \lambda)y) \leq \lambda \Phi(x) + (1 - \lambda)\Phi(y) = m, \tag{5}$$

so $\lambda x + (1 - \lambda)y$ also attains m. Hence S is convex. Now if Φ is strictly convex and $x \neq y$, then for $\lambda = 1/2$ the second inequality in (5) is strict and leads to the contradiction $m < m$. □

Let us see how to approximate the diameter of an attractor in Euclidean space. We will consider two approaches. The first one is experimental. It consists of drawing an approximation of the attractor and the assumption that the diameter of the obtained set is a good evaluation of the real diameter. This is motivated by the following proposition.

Proposition 6 *Let* $F = \{f_1, f_2, \ldots, f_N\}$ *be an IFS with attractor* $\mathcal{A}$. *Then for any non-empty bounded set* E

$$|\delta(\mathcal{A}) - \delta(F^n(E))| \leq 2 \frac{L^n}{1 - L} h(E, F(E))$$

where $L = \max_{1 \leq i \leq N} L(f_i)$.[1]

Proof. This is an immediate consequence of a result in Berger [2] which states that the function diameter is Lipschitz with constant 2. □

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[1] We recall that h is the Hausdorff metric. By $\delta(A)$ we denote the diameter of a set A.

The second approach is to compute the minimum of the function Φ. We explain in an example how this can be done. The dragon is the attractor of the IFS $\{f_1, f_2\}$, where $f_1(x,y) = (1/2x + 1/2y, -1/2x + 1/2y)$ and $f_2(x,y) = (-1/2x + 1/2y + 1, -1/2x - 1/2y)$. For a given point $u(x,y)$ we determine the value of $\Phi(u) = \max_{1\le i\le 2} \frac{d(u,f_i(u))}{1-L(f_i)}$. We know that $L(f_i) = \sqrt{\max_{1\le j\le 2}\lambda_j}$, where the $\lambda_j s$ are the eigenvalues of $A_i^T A_i$ and A_i is the matrix associated to f_i. Thus $L(f_1) = L(f_2) = \sqrt{2}/2$. By comparing $d(u, f_1(u))$ and $d(u, f_2(u))$ we find that

$$\Phi(u) = \begin{cases} \frac{d(u,f_1(u))}{1-\sqrt{2}/2} & \text{if } (x-3/4)^2 + (y+1/4)^2 \le 1/8 \\ \frac{d(u,f_2(u))}{1-\sqrt{2}/2} & \text{otherwise} \end{cases}$$

Instead of the minimization problem $\inf_{u\in I\!R^2} \Phi(u)$, we can consider the equivalent one $\inf_{u\in I\!R^2} \Phi^2(u)$. The function Φ^2 has no critical points inside or outside the circle defined by the equation $(x-3/4)^2 + (y+1/4)^2 = 1/8$. On the circle the critical points are solutions of the Lagrangian system

$$\begin{cases} \frac{\partial F(x,y)}{\partial x} = 0 \\ \frac{\partial F(x,y)}{\partial y} = 0 \\ (x-3/4)^2 + (y+1/4)^2 = 1/8 \end{cases}$$

where $F(x,y) = x^2/2 + y^2/2 - \lambda\{(x-3/4)^2 + (y+1/4)^2 - 1/8\}$. These solutions are $x = 3(\sqrt{5}/20 + 1/4)$, $y = -(\sqrt{5}/20 + 1/4)$ and $x = -3(\sqrt{5}/20 - 1/4)$, $y = (\sqrt{5}/20 - 1/4)$. By comparing the value of Φ at these points we deduce that $\inf_{u\in I\!R^2} \Phi(u) = \frac{(\sqrt{5}-1)/4}{1-\sqrt{2}/2}$. Thus the diameter of the dragon is less than $\frac{\sqrt{5}-1}{2-\sqrt{2}}$.

In the general case and for an IFS $F = \{f_1, f_2, \ldots, f_N\}$, we determine the N domains $D_1, D_2, \ldots, D_N$ defined by

$$D_i \begin{cases} \frac{d(u,f_i(u))}{1-L(f_i)} \ge \frac{d(u,f_1(u))}{1-L(f_1)} \\ \frac{d(u,f_i(u))}{1-L(f_i)} \ge \frac{d(u,f_2(u))}{1-L(f_2)} \\ \ldots \\ \frac{d(u,f_i(u))}{1-L(f_i)} \ge \frac{d(u,f_{i-1}(u))}{1-L(f_{i-1})} \\ \ldots \\ \frac{d(u,f_i(u))}{1-L(f_i)} \ge \frac{d(u,f_N(u))}{1-L(f_N)} \end{cases}$$

The function $\Phi(u)$ is equal to $\frac{d(u,f_i(u))}{1-L(f_i)}$ on D_i. We find the critical points of $\Phi^2(u)$ which is a differentiable function on every D_i. The study is made separately in the interior and on the boundary of D_i. We calculate the value of Φ at the critical points and retain those that minimize Φ.

The following proposition says that there is a special case where the diameter of the attractor is equal to the diameter of a finite set.

Proposition 7 *Let $F = \{f_1, f_2, \ldots, f_N\}$ be an affine IFS with attractor $\mathcal{A}$. Let Ω_i denote the fixed point of f_i, and let $\mathcal{O}$ be the convex hull of $\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$. If $f_j(\Omega_i) \in \mathcal{O}$ for all $i, j \in \{1, 2, \ldots, N\}$, then the diameter of $\mathcal{A}$ is equal to the diameter of $\mathcal{O}$.*

Proof. Since all the fixed points Ω_i are contained in the attractor, we have $\delta(\mathcal{O}) \leq \delta(\mathcal{A})$. The condition $f_j(\Omega_i) \in \mathcal{O}$ implies that $f_j(\mathcal{O}) \subset \mathcal{O}$. Thus $\mathcal{A} \subset \mathcal{O}$ which shows that $\delta(\mathcal{O}) \geq \delta(\mathcal{A})$. □

Example *The diameter of the Sierpinski gasket is equal to the diameter of the triangle whose vertices are the fixed points of the three contractions.*

The computation of the diameter of a finite set in Euclidean space is a common problem in computational geometry. Efficient algorithms are given in Preparata and Shamos [8].

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