Confidence-consistency driven group decision making approach with incomplete reciprocal intuitionistic preference relations

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Abstract

Intuitionistic preference relations constitute a flexible and simple representation format of experts’ preference on a set of alternative options, while at the same time allowing to accommodate degrees of hesitation inherent to all decision making processes. In comparison with fuzzy preference relations, the use of intuitionistic fuzzy preference relations in decision making is limited, which is mainly due to the computational complexity associated to using membership degree, non-membership degree and hesitation degree to model experts’ subjective preferences. In this paper, the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations are proved to be mathematically isomorphic. This result can be exploited to use methodologies developed for fuzzy preference relations to the case of intuitionistic fuzzy preference relations and, ultimately, to overcome the computation complexity mentioned above and to extend the use of reciprocal intuitionistic fuzzy preference relations in decision making. In particular, in this paper, this isomorphic equivalence is used to address the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making by developing a consistency driven estimation procedure via the corresponding equivalent incomplete asymmetric fuzzy preference relation procedure. Additionally, the hesitancy degree of the reciprocal intuitionistic fuzzy preference relation is used to introduce the concept of expert’s confidence from which a group decision making procedure, based on a new aggregation operator that takes into account not only the experts’ consistency but also their confidence degree towards the opinion provided, is proposed.

Keywords: Group decision making, Uncertainty, Incomplete information, Intuitionistic fuzzy preference relations, Asymmetric fuzzy preference relations, Uninorm.

1. Introduction

Intuitionistic fuzzy preference relations are based on the concept of intuitionistic fuzzy set that Atanassov introduced in [3] as an extension of the concept of fuzzy set. Due to its flexibility in handling vagueness/uncertainty, intuitionistic fuzzy set theory [4] has been extensively used in many areas, such as virtual medical diagnosis [11], pattern recognition [26], clustering analysis [30] and decision making [23, 27–29]. For example in [10], Fujita et al. propose to model the user cognitive behaviour on mental cloning-based software using intuitionistic fuzzy sets.

Much research has been carried out in decision making with preferences modelled using fuzzy relations in comparison to using intuitionistic fuzzy relations. This is mainly to the longer existence of the former representation format of preferences in comparison to the second

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one. However, an additional cause for the lesser use of intuitionistic fuzzy preference relations in decision making is the increase computational complexity associated to the use of membership degree, non-membership degree and hesitation degree to model experts’ subjective preferences. Notice that intuitionistic fuzzy preference relations are usually assumed to be reciprocal (Section 2).

A first objective of this paper is to prove the mathematical equivalence between the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations. This result can thus be exploited to use methodologies developed for fuzzy preference relations to the case of intuitionistic fuzzy preference relations and, ultimately, to extend the use of reciprocal intuitionistic fuzzy preference relations in decision making and to overcome the computation complexity mentioned above. In other word, this result will allow to take advantage of mature and well defined methodologies developed for fuzzy preference relations while leveraging the flexibility of reciprocal intuitionistic fuzzy preference relations to model vagueness/uncertainty. Indeed, an issue that can be addressed using the mentioned equivalence is the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making.

Incomplete information as a result from the incapability of experts to provide complete information about their preferences [7, 14] may happen more frequently than expected due to different reasons such as: experts not having a precise or sufficient level of knowledge of part of the problem, lack of time, difficulty to distinguish up to which degree one preference is better than other, or conflicting between alternatives, among others. In the literature, different approaches to deal with missing or incomplete information have been extensively studied for the case of using fuzzy preference relations as the representation format of preferences [25]. Most of the existing approaches are based on the selection of an appropriate methodology to ‘build’ the matrix, and/or to assign importance values to experts based not on the amount of information provided but on how consistent the information provided is [1, 8, 14, 18, 28].

The case of incomplete intuitionistic fuzzy preference relations has been addressed in literature in [28, 29], where the above mentioned methodology to estimate missing information driven by the consistency was adopted. The main difference between both approaches resides in the way consistency of reciprocal intuitionistic fuzzy preference relations was modelled. On the one hand, in [29] a straight forward transposition of the multiplicative consistency property for fuzzy preference relations was proposed for the case of reciprocal intuitionistic fuzzy preference relations, which has been later proved to be incorrect [28], and publicly acknowledged by the authors that proposed it [31]. On the other hand, in [28] the concept of multiplicative consistency for reciprocal intuitionistic fuzzy preference relations was derived by formally extending the multiplicative transitivity property for fuzzy preference relations via the use of both the Extension Principle and Representation Theorem [35]. In this contribution, though, a different approach to incomplete reciprocal intuitionistic fuzzy preference relations is presented based on the aforementioned equivalence between the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations. The main advantage of the approach put forward here is that the isomorphic relation between reciprocal intuitionistic fuzzy preference relations and asymmetric fuzzy preference relations makes superfluous both the extension principle and the representation theorem that were required in [28], as well as being less computationally complex because there is no need to split the reciprocal intuitionistic fuzzy preference relations into two reciprocal fuzzy preference relations but one single asymmetric fuzzy preference relation.

A second objective of this paper is to develop a fuse approach of the information provided by the experts taking into account the confidence level of each expert in his/her own opinion, which is intrinsically connected to the information he/she provides [12], and which in the case of reciprocal intuitionistic fuzzy preference relations is linked to the associated hesitancy
function. Obviously, the more confident the expert feels about his/her opinion the more relevant the opinion can be considered, and thus more importance should be allocated to it. This can be achieved in the aggregation phase of a group decision making model by implementing an appropriate confidence and consistency based induced ordered weighted average to compute the collective preferences [6, 14, 28].

The rest of the paper is set out as follows: Section 2 presents the main mathematical frameworks for representing preferences of interest, while Section 3 deals with the concept of consistency of fuzzy preference relations as needed throughout the rest of the paper. Section 4 demonstrates the mathematical equivalence between the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations, which is used in Section 5 to present a methodology to estimate missing values of reciprocal intuitionistic fuzzy preference relations. The hesitancy function is proposed in a confidence-consistency driven group decision making approach with incomplete reciprocal intuitionistic fuzzy preference relations whose application is illustrated with an example in in Section 6. Finally, Section 7 includes an analysis of the proposed group decision making model, including some future work and draws conclusions.

2. Preference Relations in Decision Making

In any decision making problem, once the set of feasible alternatives \((X)\) is identified, experts are called to express their opinions or preferences on such set. Different preference elicitation methods were compared in [19], concluding that pairwise comparison methods are more accurate than non-pairwise methods since it allows the expert to focus only in two alternatives at a time. A comparison of two alternatives of \(X\) by an expert can lead to the preference of one alternative to the other or to a state of indifference between them. Obviously, there is the possibility of an expert being unable to compare them. Two main mathematical models based on the concept of preference relation can be used in this context. In the first one, a preference relation is defined for each one of the above three possible preference states (preference, indifference, incomparability), which is known as a preference structure on the set of alternatives. The second one integrates the three possible preference states into a single preference relation. In this paper, we focus on the second one as per the following definition:

**Definition 1 (Preference Relation).** A preference relation \(P\) on a set \(X\) is a binary relation \(\mu_P : X \times X \to D\), where \(D\) is the domain of representation of preference degrees provided by the decision maker.

A preference relation \(P\) may be conveniently represented by a matrix \(P = (p_{ij})\) of dimension \(\text{card}(X)\), with \(p_{ij} = \mu_P(x_i, x_j)\) being interpreted as the degree or intensity of preference of alternative \(x_i\) over \(x_j\). The elements of \(P\) can be of a numeric or linguistic nature, i.e., could represent numeric or linguistic preferences, respectively [20]. The main types of numeric preference relations used in decision making are: crisp preference relations, additive preference relations, multiplicative preference relations, interval-valued preference relations and intuitionistic preference relations. In this contribution we are going to focus on the reciprocal intuitionistic fuzzy preference relations and their equivalence to a subclass of asymmetric fuzzy preference relations.

2.1. Fuzzy Set and Fuzzy Preference Relation

**Definition 2 (Fuzzy Set).** Let \(U\) be a universal set defined in a specific problem, with a generic element denoted by \(x\). A fuzzy set \(X\) in \(U\) is a set of ordered pairs:

\[
X = \{(x, \mu_X(x))| x \in U\}
\]
where $\mu_X: U \rightarrow [0, 1]$ is called the membership function of $A$ and $\mu_X(x)$ represents the degree of membership of the element $x$ in $X$.

Notice that the degree of non-membership of the element $x$ in $X$ is here defined as $\nu_X(x) = 1 - \mu_X(x)$. Thus, $\mu_X(x) + \nu_X(x) = 1$.

**Definition 3** (Fuzzy Preference Relation). A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives $X$ is a fuzzy relation in $X \times X$ that is characterised by a membership function $\mu_R: X \times X \rightarrow [0, 1]$ with the following interpretation:

- $r_{ij} = 1$ indicates the maximum degree of preference for $x_i$ over $x_j$
- $r_{ij} \in ]0, 1]$ indicates a definite preference for $x_i$ over $x_j$
- $r_{ij} = 1/2$ indicates indifference between $x_i$ and $x_j$

When

$$r_{ij} + r_{ji} = 1 \ \forall i, j \in \{1, \ldots, n\}$$

is imposed we have a reciprocal fuzzy preference relation.

### 2.2. Intuitionistic fuzzy set and Intuitionistic fuzzy preference relation

**Definition 4** (Intuitionistic Fuzzy Set). An intuitionistic fuzzy set $X$ over a universe of discourse $U$ is given by

$$X = \left\{ (x, (\mu_X(x), \nu_X(x))) \mid x \in U \right\}$$

where $\mu_X: U \rightarrow [0, 1]$ and $\nu_X: U \rightarrow [0, 1]$ verify

$$0 \leq \mu_X(x) + \nu_X(x) \leq 1 \ \forall x \in U.$$ 

$\mu_X(x)$ and $\nu_X(x)$ degree of membership and degree of non-membership of $x$ to $X$.

An intuitionistic fuzzy set becomes a fuzzy set when $\mu_X(x) = 1 - \nu_X(x) \ \forall x \in U$. However, when there exists at least a value $x \in U$ such that $\mu_X(x) < 1 - \nu_X(x)$, an extra parameter has to be taken into account when working with intuitionistic fuzzy sets: the hesitancy degree, $\tau_X(x) = 1 - \mu_X(x) - \nu_X(x)$, that represents the amount of lacking information in determining the membership of $x$ to $X$. If the hesitation degree is zero, the reciprocal relationship between membership and non-membership makes the latter one unnecessary in the formulation as it can be derived from the former.

In [23], Szmidt and Kacprzyk defined the intuitionistic fuzzy preference relation as a generalisation of the concept of fuzzy preference relation, which is adopted in the following definition.

**Definition 5** (Intuitionistic Fuzzy Preference Relation). An intuitionistic fuzzy preference relation $B$ on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_B: X \times X \rightarrow [0, 1]$ and a non-membership function $\nu_B: X \times X \rightarrow [0, 1]$ such that

$$0 \leq \mu_B(x_i, x_j) + \nu_B(x_i, x_j) \leq 1 \ \forall (x_i, x_j) \in X \times X.$$ 

with $\mu_B(x_i, x_j) = \mu_{ij}$ interpreted as the certainty degree up to which $x_i$ is preferred to $x_j$; and $\nu_B(x_i, x_j) = \nu_{ij}$ interpreted as the certainty degree up to which $x_i$ is non-preferred to $x_j$. 
An intuitionistic fuzzy preference relation can also be represented by a matrix $B = (b_{ij})$ with $b_{ij} = (\mu_{ij}, \nu_{ij}) \forall i, j = 1, 2, \ldots, n$. Notice that when the hesitancy function is the null function we have that $\mu_{ij} + \nu_{ij} = 1 \forall i, j$, and therefore the intuitionistic fuzzy preference relation $B = (b_{ij})$ is mathematically equivalent to the reciprocal fuzzy preference relation $(\mu_{ij})$, i.e. $B = (\mu_{ij})$. An intuitionistic fuzzy preference relation is referred to as reciprocal when the following additional conditions are imposed:

- $\mu_{ii} = \nu_{ii} = 0.5 \forall i \in \{1, \ldots, n\}$.
- $\mu_{ji} = \nu_{ij} \forall i, j \in \{1, \ldots, n\}$.

Notice that when the hesitancy degree function is the null function in a reciprocal intuitionistic fuzzy preference relation, $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, then it is $\mu_{ij} + \nu_{ij} = 1 \forall i, j$, and $B$ is mathematically equivalent to the reciprocal fuzzy preference relation $R = (r_{ij}) = (\mu_{ij})$.

### 3. Consistency of fuzzy preference relations

Consistency of fuzzy preference relations has been modeled using the notion of transitivity in the pairwise comparison among any three alternatives: if $x_i$ is preferred to $x_j$ $(x_i \succ x_j)$ and this one to $x_k$ $(x_j \succ x_k)$ then alternative $x_i$ should be preferred to $x_k$ $(x_i \succ x_k)$, which is normally referred to as *weak stochastic transitivity* [5]. Any property that guarantees the transitivity of the preferences is called a consistency property. Clearly, the lack of consistency in decision making can lead to inconsistent conclusions; that is why it is crucial to study conditions under which consistency is satisfied [21].

Different properties or conditions have been suggested as rational conditions to be verified by a consistent fuzzy preference relation [5, 15]: triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, additive transitivity, and multiplicative transitivity. The last two properties, proposed by Tanino in [24], are the most widely used in the context of incomplete information [5].

**Definition 6** (Additive transitivity [24]). A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives $X$ is additive transitive if and only if

$$(r_{ij} - 0.5) + (r_{jk} - 0.5) = r_{ik} - 0.5 \forall i, j, k = 1, 2, \ldots, n$$

Although equivalent to Saaty’s consistency property for multiplicative preference relations [15], additive transitivity is in conflict with the $[0, 1]$ scale used for providing the preference values and therefore it is not appropriate to model consistency of fuzzy preference relations [5]. Tanino also proposed the following alternative transitivity property:

**Definition 7** (Multiplicative transitivity [24]). A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives $X$ is multiplicative transitive if and only if

$$r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji} \forall i, k, j \in \{1, 2, \ldots, n\}$$

Chiclana et al. in [5] propose the modeling of the cardinal consistency of reciprocal fuzzy preference relations via a functional equation, and they proved that when such a function is almost continuous and monotonic (increasing) then it must be a representable uninorm. Cardinal consistency with the conjunctive representable cross ratio uninorm

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\} \\ \frac{xy}{xy + (1 - x)(1 - y)}, & \text{otherwise} \end{cases}$$

5
was proved equivalent to Tanino’s multiplicative transitivity property, and because any two representable uninorms are order isomorphic, it was concluded that multiplicative transitivity is the most appropriate property to model consistency of reciprocal fuzzy preference relations. It is worth reminding that multiplicative transitivity property extends weak stochastic transitivity, and therefore extends the classical transitivity property of crisp preference relations. This is why we refer to this property as the multiplicative consistency property.

Multiplicative consistency property (1) can be used to estimate the preference value between a pair of alternatives \((x_i, x_j)\) with \((i < j)\) using another different intermediate alternative \(x_k\) \((k \neq i, j)\) as follows:

\[
mr_{ij}^k = \frac{r_{ik} \cdot r_{kj} \cdot r_{ji}}{r_{jk} \cdot r_{ki}}
\]  

(3)
as long as the denominator is not zero. We call \(mr_{ij}^k\) the partially multiplicative transitivity based estimated fuzzy preference value of the pair of alternatives \((x_i, x_j)\) obtained using the intermediate alternative \(x_k\).

Notice that expression (1) is always true when two of the three subindexes are equal. Let \(k = i\) and \(r_{ji} \neq 0\) then \(mr_{ij}^i = r_{ij}\), while if \(r_{ij} \neq 0\) then \(mr_{ij}^i = r_{ji}\). Because \(r_{ji} = 1 - r_{ij}\), then we have that: \(r_{ji} \neq 0\) if and only if \(r_{ij} \neq 1\). Thus, if \(k = i\) and \((r_{ij}, r_{ji}) \notin \{(1, 0), (0, 1)\}\) we have \(mr_{ij}^i = r_{ij}\) and \(mr_{ij}^i = r_{ji}\). A similar reasoning and conclusion is obtained when \(k = j\). Summarising, although it is possible to obtain the multiplicative transitivity based estimated fuzzy preference value of the pair of alternatives \((x_i, x_j)\) when \(k \in \{i, j\}\) and \((r_{ij}, r_{ji}) \notin \{(1, 0), (0, 1)\}\), it is also true that there is no indirect estimation process as described above. Furthermore, when the fuzzy preference value \(r_{ij}\) is unknown its estimation will automatically require that \(k \notin \{i, j\}\). Finally, when \(i = j\) we have by definition that \(r_{ii} = 0.5\) and we would have \(mr_{ii}^k = r_{ii}\) whenever \(r_{ik} \notin \{(0, 1), (1, 0)\}\). Thus, this last case will not be relevant when having incomplete information, and it is also not assumed from now on.

The average of all possible partially multiplicative transitivity based estimated values of the pair of alternatives \((x_i, x_j)\) can be interpreted as their global multiplicative transitivity based estimated value

\[
mr_{ij} = \frac{\sum_{k \in R_{ij}^0} mr_{ij}^k}{\#R_{ij}^0};
\]

where \(R_{ij}^0 = \{k \neq i, j | (r_{ik}, r_{kj}) \notin R_{ij}^1\}\), \(R_{ij}^1 = \{(1, 0), (0, 1)\}\), and \(\#R_{ij}^0\) is the cardinality of \(R_{ij}^0\). Therefore, given a fuzzy preference relation, \(R = (r_{ij})\), the following multiplicative transitivity based fuzzy preference relation, \(MR = (mr_{ij})\), can be constructed. Notice that when a fuzzy preference relation \(R = (r_{ij})\) is multiplicative transitive then \(R = MR\). Indeed, if \(R\) is multiplicative transitive then (1) holds \(\forall i, j, k\). In particular, we have

\[
r_{ij} = \frac{r_{ik} \cdot r_{kj} \cdot r_{ji}}{r_{jk} \cdot r_{ki}};
\]

whenever \(k \in R_{ij}^0\). Consequently, \(mr_{ij}^k = r_{ij}\) for all \(i, j\) and \(k \in R_{ij}^0\), which proves that \(r_{ij} = mr_{ij}\) for all \(i, j\). A fuzzy preference relation \(R\) will be referred to as multiplicative consistent from now on when \(R = MR\).

**Definition 8 (Multiplicative Consistency).** A fuzzy preference relation \(R = (r_{ij})\) is multiplicative consistent if and only if \(R = MR\).

The similarity between the values \(r_{ij}\) and \(mr_{ij}\) is proposed to be used in measuring the level of consistency of a fuzzy preference relation at its three different levels: pair of alternatives, alternatives and relation [14]:
Level 1. Consistency Index of pair of alternatives.

\[ CL_{ij} = 1 - d(r_{ij}, mr_{ij}) \quad \forall i, j. \]

Here \( d(r_{ij}, mr_{ij}) \) represents the distance between the values \( r_{ij} \) and \( mr_{ij} \). Obviously, the higher the value of \( CL_{ij} \) the more consistent is \( r_{ij} \) with respect to the rest of the preference values involving alternatives \( x_i \) (row \( i \) of the fuzzy preference relation) and \( x_j \) (column \( j \) of the fuzzy preference relation).

Level 2. Consistency Level of alternatives.

\[ CL_i = \frac{\sum_{j=1; \ i \neq j}^{n} CL_{ij}}{n-1}. \]

Level 3. Consistency Level of a fuzzy preference relation.

\[ CL = \frac{\sum_{i=1}^{n} CI_i}{n}. \]

The following result characterises multiplicative consistency of a fuzzy preference relation using its corresponding consistency level.

Proposition 1. A fuzzy preference relation \( R \) is multiplicative consistent if and only if \( CL = 1 \).

Proof.

1. \( R \) is multiplicative consistent \( \implies \) \( CL = 1 \). Because \( R \) is assumed to be multiplicative consistent the Definition 8 applies and we have that \( R = MR \), which means that \( d(r_{ij}, mr_{ij}) = 0 \ \forall i, j \). Consequently, \( CL = 1 \).

2. \( CL = 1 \implies \) \( R \) is multiplicative consistent. If \( CL = 1 \) then \( \sum_{i,j=1; i \neq j}^{n} CL_{ij} = n \times (n-1) \). We have that \( CL_{ij} \in [0, 1] \) and therefore it is \( CL_{ij} = 1 \ \forall i \neq j \) otherwise it would be \( \sum_{i,j=1; i \neq j}^{n} CL_{ij} < n \times (n-1) \). Therefore we have that \( CL = 1 \) if and only if \( r_{ij} = mr_{ij} \ \forall i \neq j \). Finally, when \( i = j \) we have \( mr_{ii}^k = r_{ii} = 0.5 \) whenever \( r_{ik} \notin \{(0, 1), (1, 0)\} \), and therefore \( mr_{ii} = 0.5 \). Thus, we have that \( r_{ij} = mr_{ij} \ \forall i, j \), i.e. \( R = MR \), and conclude that \( R \) is multiplicative consistent.

4. Reciprocal intuitionistic fuzzy preference relations and asymmetric fuzzy preference relations

This section proves the equivalence between the set of asymmetric fuzzy preference relations and the set of reciprocal intuitionistic fuzzy preference relations, and will provide the isomorphism to derive an asymmetric fuzzy preference relation given a reciprocal intuitionistic fuzzy preference relation. This result is exploited Section 5 to tackle the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making by developing a consistency driven estimation procedure via the corresponding equivalent incomplete asymmetric fuzzy preference relation.
Given an intuitionistic fuzzy preference relation $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, the constraint $0 \leq \mu_{ij} + \nu_{ij} \leq 1$ implies that an element of $B$ can be represented by a point in the lower half unit square area as shaded in Fig. 1(a). Notice that in the case of being the intuitionistic fuzzy preference relation $B$ reciprocal, given the element $\langle \mu_{ij}, \nu_{ij} \rangle$ then we have that $\langle \mu_{ji}, \nu_{ji} \rangle = \langle \nu_{ij}, \mu_{ij} \rangle$, i.e. $\langle \mu_{ji}, \nu_{ji} \rangle$ is the mirror image of $\langle \mu_{ij}, \nu_{ij} \rangle$ with respect to the line $\mu_B = \nu_B$ as illustrated in Fig. 1(b). Consequently, a reciprocal intuitionistic fuzzy preference relation $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$

\[
B = \begin{pmatrix}
\langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \cdots & \langle \mu_{1i}, \nu_{1i} \rangle & \cdots & \langle \mu_{1n}, \nu_{1n} \rangle \\
\langle \mu_{21}, \nu_{21} \rangle & \langle \mu_{22}, \nu_{22} \rangle & \cdots & \langle \mu_{2i}, \nu_{2i} \rangle & \cdots & \langle \mu_{2n}, \nu_{2n} \rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\langle \mu_{i1}, \nu_{i1} \rangle & \langle \mu_{i2}, \nu_{i2} \rangle & \cdots & \langle \mu_{ii}, \nu_{ii} \rangle & \cdots & \langle \mu_{in}, \nu_{in} \rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\langle \mu_{n1}, \nu_{n1} \rangle & \langle \mu_{n2}, \nu_{n2} \rangle & \cdots & \langle \mu_{ni}, \nu_{ni} \rangle & \cdots & \langle \mu_{nn}, \nu_{nn} \rangle
\end{pmatrix}
\]

can be completely characterised using just its upper triangular part

\[
UB = \begin{pmatrix}
\langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \cdots & \langle \mu_{1i}, \nu_{1i} \rangle & \cdots & \langle \mu_{1n}, \nu_{1n} \rangle \\
\langle \mu_{21}, \nu_{21} \rangle & \langle \mu_{22}, \nu_{22} \rangle & \cdots & \langle \mu_{2i}, \nu_{2i} \rangle & \cdots & \langle \mu_{2n}, \nu_{2n} \rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\langle \mu_{i1}, \nu_{i1} \rangle & \langle \mu_{i2}, \nu_{i2} \rangle & \cdots & \langle \mu_{ii}, \nu_{ii} \rangle & \cdots & \langle \mu_{in}, \nu_{in} \rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\langle \mu_{n1}, \nu_{n1} \rangle & \langle \mu_{n2}, \nu_{n2} \rangle & \cdots & \langle \mu_{ni}, \nu_{ni} \rangle & \cdots & \langle \mu_{nn}, \nu_{nn} \rangle
\end{pmatrix}
\]

and this one can be represented equivalently as the following fuzzy preference relation

\[
R = \begin{pmatrix}
\mu_{11} & \mu_{12} & \cdots & \mu_{1i} & \cdots & \mu_{1n} \\
\nu_{12} & \nu_{12} & \cdots & \nu_{1i} & \cdots & \nu_{1n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\nu_{1i} & \nu_{2i} & \cdots & \mu_{ii} & \cdots & \mu_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\nu_{1n} & \nu_{2n} & \cdots & \nu_{in} & \cdots & \mu_{nn}
\end{pmatrix}
\]
Because \( \nu_{ij} = \mu_{ji} \) then we have that
\[
R = \begin{pmatrix}
\mu_{11} & \mu_{12} & \cdots & \mu_{1i} & \cdots & \mu_{1n} \\
\mu_{21} & \mu_{22} & \cdots & \mu_{2i} & \cdots & \mu_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mu_{i1} & \mu_{i2} & \cdots & \mu_{ii} & \cdots & \mu_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mu_{n1} & \mu_{n2} & \cdots & \mu_{ni} & \cdots & \mu_{nn} 
\end{pmatrix}.
\]
This is illustrated in the following diagram:

\[
\begin{array}{c}
B \\
\rightarrow \\
UB \\
\downarrow \\
R
\end{array}
\]

In the following, we formalise the above relationship. Let denote with \( \mathcal{B} \) the set of reciprocal intuitionistic fuzzy preference relations:
\[
\mathcal{B} = \left\{ B = (b_{ij}) | \forall i, j : b_{ij} = (\mu_{ij}, \nu_{ij}), \mu_{ij}, \nu_{ij} \in [0, 1], \mu_{ii} = \nu_{ii} = 0.5, \mu_{ij} = \nu_{ji}, 0 \leq \mu_{ij} + \nu_{ij} \leq 1 \right\}
\]
and with \( \mathcal{R} \) the set of fuzzy preference relations
\[
\mathcal{R} = \left\{ R = (r_{ij}) | \forall i, j : r_{ij} \in [0, 1] \right\}
\]
Let \( f : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be the following function \( f(x_1, x_2) = x_1 \). We can define the following mapping, \( F : \mathcal{B} \rightarrow \mathcal{R} \), between the set of reciprocal intuitionistic fuzzy preference relations, \( \mathcal{B} \), and the set of fuzzy preference relations, \( \mathcal{R} \),
\[
R = F(B) = (f(b_{ij})) = (\mu_{ij}).
\]
The following properties can be proved:

**Proposition 2.** Function \( F \) is well defined, i.e. given \( B \in \mathcal{B} \) it is true that \( f(B) \in \mathcal{R} \).

*Proof.* It is obvious and it is omitted. \( \square \)

**Proposition 3.** Function \( F \) is an injection.

*Proof.* Let \( B_1 = (b^1_{ij}) \) and \( B_2 = (b^2_{ij}) \) two reciprocal intuitionistic fuzzy preference relation such that \( F(B_1) = F(B_2) \). Then we have that
\[
f(b^1_{ij}) = f(b^2_{ij}) \iff \mu^1_{ij} = \mu^2_{ij} \forall i, j.
\]
Because \( \mu^1_{ij} = \nu^1_{ji} \) and \( \mu^2_{ij} = \nu^2_{ji} \) then it is obvious that
\[
\nu^1_{ij} = \nu^2_{ji} \forall i, j.
\]
Therefore we have that
\[
b^1_{ij} = (\mu^1_{ij}, \nu^1_{ij}) = (\mu^2_{ij}, \nu^2_{ij}) = b^2_{ij} \forall i, j.
\]
Consequently, it is concluded that \( B_1 = B_2 \). \( \square \)
For function $F$ to be a surjection, the following needs to be verified:

$$\forall R \in \mathcal{R} \ \exists B \in \mathcal{B}: \ F(B) = R.$$ 

However, by definition of $\mathcal{B}$ and $F$ we have that $R = (r_{ij}) = (\mu_{ij})$ verifies:

$$0 \leq r_{ij} + r_{ji} = \mu_{ij} + \mu_{ji} = \mu_{ij} + \nu_{ij} \leq 1.$$ 

Thus $R$ is an asymmetric fuzzy preference relation [9]. This proves that the range of function $F$ is the subset of fuzzy preference relations that are asymmetric. In other words:

**Proposition 4.** The range of function function $F$ is the set of asymmetric fuzzy preference relations, i.e. function $F$ is not a surjection.

Putting together these results we have:

**Theorem 1.** The set of reciprocal intuitionistic fuzzy preference relations is isomorphic to the set of asymmetric fuzzy preference relations.

Asymmetric fuzzy preference relations guarantee that when $p_{ij} \geq 0.5$ then $p_{ji} \leq 0.5$. In preference modelling it guarantees that both $p_{ij}$ and $p_{ji}$ cannot be high at the same time. In other words, an asymmetric fuzzy preference relation guarantees that when an alternative $x$ is preferred to another alternative $y$, then alternative $y$ is not preferred to alternative $x$.

To conclude this section, we notice that when $B \in \mathcal{B}$ has hesitancy degree always zero then we have that:

$$\mu_{ij} + \nu_{ij} = 1 \ \forall i,j.$$ 

(4)

In these cases, $F(B) = R$ is also reciprocal, i.e. $r_{ij} + r_{ji} = 1 \forall i,j$. The proof of this is quite simple as we have the following:

$$\forall i,j : \ r_{ij} = f(b_{ij}) = \mu_{ij} \ \land \ r_{ji} = f(b_{ji}) = \mu_{ji}.$$ 

However, because $B$ is reciprocal then we have that $\mu_{ji} = \nu_{ij} \ \forall i,j$ and consequently applying (4) it is:

$$r_{ij} + r_{ji} = \mu_{ij} + \mu_{ji} = \mu_{ij} + \nu_{ij} = 1 \ \forall i,j.$$ 

Thus, as previously mentioned in Section 2 the subset of reciprocal fuzzy preference relations $\{ B \in \mathcal{B} | \mu_{ij} + \nu_{ij} = 1 \ \forall i,j \}$, highlighted in red in Fig. 1, is invariant under function $F$, i.e. function $F$ is the identity function when reduced to the subset of reciprocal fuzzy preference relations.

5. Incomplete reciprocal intuitionistic fuzzy preference relations

The previous section main result allows to solve problems associated to reciprocal intuitionistic fuzzy preference relations by solving the corresponding problem to their equivalent asymmetric fuzzy preference relations. Thus, the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making can be tackled by using the estimation procedure presented in Section 3 to the corresponding equivalent incomplete asymmetric fuzzy preference relations. Before doing this, we present the formal definition of the concept of incomplete preference relation [14]:

**Definition 9.** A function $g : X \rightarrow Y$ is partial when not every element in the set $X$ necessarily maps to an element in the set $Y$. When every element from the set $X$ maps to one element of the set $Y$ then we have a total function.
Definition 10. A preference relation on a set of alternatives with a partial membership function is an incomplete preference relation.

It is assumed that for incomplete reciprocal intuitionistic fuzzy preference relations, given a pair of alternatives \((x_i, x_j)\) for which \(b_{ij}\) is not known, both membership and non-memberships will be unknown. Due to reciprocity, we have that if \(b_{ij}\) is not known then \(b_{ji}\) is also not known. In general the letter \(x\) will be used when a particular entry of an incomplete reciprocal intuitionistic fuzzy preference relation is unknown/missing. Thus, if \(B\) is an incomplete reciprocal intuitionistic fuzzy preference relation, then \(R = F(B)\) will be an incomplete asymmetric fuzzy preference relation. However, the missing preference value \(r_{ij} (i \neq j)\) cannot be partially estimated, using an intermediate alternative \(x_k\), via expression (1) because \(r_{ij}\) is also unknown. As we have already mentioned, under reciprocity, Tanino’s multiplicative transitivity property (1) cab be rewritten as (2). Thus the missing preference value \(r_{ij}(i \neq j)\) can be partially estimated, using an intermediate alternative \(x_k\), with the value:

\[
\text{cr}^k_{ij} = \left\{ \begin{array}{ll}
0, & (r_{ik}, r_{kj}) \in \{(0, 1), (1, 0)\} \\
\frac{r_{ik} \cdot r_{kj}}{r_{ik} \cdot r_{kj} + (1 - r_{ik}) \cdot (1 - r_{kj})}, & \text{Otherwise.} 
\end{array} \right.
\]

The following notation is introduced:

\[
\begin{aligned}
A &= \{(i, j) \mid i, j \in \{1, \ldots, n\} \land i \neq j\}; \\
MV &= \{(i, j) \mid r_{ij} \text{ is unknown, } (i, j) \in A\}; \\
EV &= A \setminus MV.
\end{aligned}
\]

\(MV\) is the set of pairs of different alternatives for which the fuzzy preference degree is unknown or missing; \(EV\) is the set of pairs of different alternatives with known fuzzy preference values. The global multiplicative transitivity based estimated value, \(\text{cr}_{ij}\), is defined as follows:

\[
\text{cr}_{ij} = \frac{\sum_{k \in R_{ij}^{01}} \text{cr}^k_{ij}}{\#R_{ij}^{01}}
\]

where \(H_{ij}^{01} = \{k \in R_{ij}^{01} | (i, j) \in MV \land (i, k), (k, j) \in EV\}\).

The iterative procedure to complete reciprocal fuzzy preference relations developed in [14] can be applied to complete \(R\) and, consequently, to complete the incomplete reciprocal intuitionistic fuzzy preference relation \(B\) as the following example illustrates. Notice that the cases when an incomplete fuzzy preference relation cannot be successfully completed are reduced to those when no preference values involving a particular alternative are known. This is the case when a whole row or column is completely missing. Therefore the general sufficient condition for an incomplete fuzzy preference relation to be completed is that a set of \(n - 1\) non-leading diagonal preference values with each one of the alternatives compared at least once is known.

Example 1. Let \(X = \{x_1, x_2, x_3, x_4\}\) be a set of alternatives evaluated by a decision maker against a particular criterion using the following incomplete reciprocal intuitionistic fuzzy preference relation [28]:

\[
B = \begin{pmatrix}
\langle 0.50, 0.50 \rangle & \langle 0.40, 0.30 \rangle & x & x \\
\langle 0.30, 0.40 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.50, 0.40 \rangle & x \\
x & \langle 0.40, 0.50 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.30, 0.40 \rangle \\
x & x & \langle 0.40, 0.30 \rangle & \langle 0.50, 0.50 \rangle
\end{pmatrix}
\]

The associated incomplete asymmetric fuzzy preference relation is:
\[ R = \begin{pmatrix} 0.5 & 0.4 & - & - \\ 0.3 & 0.5 & 0.5 & - \\ - & 0.4 & 0.5 & 0.3 \\ - & - & 0.4 & 0.5 \end{pmatrix} \]

**Step 1:** The set of elements that can be estimated at this stage are:

\[ EMV_1 = \{(1, 3), (2, 4), (3, 1), (4, 2)\}. \]

Notice that (1, 4) cannot be estimated at this step. Indeed, the estimation of element (1, 4) requires that at least one of the following pairs of preference values are known: \{(1, 2), (2, 4)\}, \{(1, 3), (3, 4)\}. However, the preference values for (2, 4) and (1, 3) are unknown. The same applies to (4, 1), which cannot be estimated at this step because the preference values for (4, 2) and (3, 1) are unknown.

The computation of the estimated values \( cr_{13} \) and \( cr_{31} \) are given below, which make use of intermediate and different alternatives \( k \) so that the chain of preference values \((1, k)\) and \((k, 3)\) are known. The only intermediate alternative to use at this step is \( k = 2 \), for which we have (rounding to 2 decimal places):

\[ cr_{13}^2 = \frac{r_{12} \cdot r_{23}}{r_{12} \cdot r_{23} + (1 - r_{12}) \cdot (1 - r_{23})} = \frac{0.4 \cdot 0.5}{0.4 \cdot 0.5 + 0.6 \cdot 0.5} = 0.4, \]

and

\[ cr_{31}^2 = \frac{r_{32} \cdot r_{21}}{r_{32} \cdot r_{21} + (1 - r_{32}) \cdot (1 - r_{21})} = \frac{0.4 \cdot 0.3}{0.4 \cdot 0.3 + 0.6 \cdot 0.7} = 0.22. \]

The computation of the estimated values \( cr_{24} \) and \( cr_{42} \) is done using intermediate alternative \( k = 3 \):

\[ cr_{24}^3 = \frac{r_{24} \cdot r_{34}}{r_{24} \cdot r_{34} + (1 - r_{24}) \cdot (1 - r_{34})} = \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.5 \cdot 0.7} = 0.3, \]

and

\[ cr_{42}^3 = \frac{r_{43} \cdot r_{32}}{r_{43} \cdot r_{32} + (1 - r_{43}) \cdot (1 - r_{32})} = \frac{0.4 \cdot 0.4}{0.4 \cdot 0.4 + 0.6 \cdot 0.6} = 0.31. \]

After the estimation process is applied, we have:

\[ R = \begin{pmatrix} 0.5 & 0.4 & 0.4 & - \\ 0.3 & 0.5 & 0.5 & 0.3 \\ 0.22 & 0.4 & 0.5 & 0.3 \\ - & 0.31 & 0.4 & 0.5 \end{pmatrix} \]

**Step 2:** The remaining unknown elements can be estimated at this stage, \( EMV_2 = \{(1, 4), (4, 1)\}. \)

We elaborate the computation process of the estimated value for \( cr_{14} \) (rounding to 2 decimal places):

\[ cr_{14}^2 = \frac{r_{12} \cdot r_{24}}{r_{12} \cdot c_{24} + (1 - c_{12}) \cdot (1 - c_{24})} = \frac{0.4 \cdot 0.3}{0.4 \cdot 0.3 + 0.6 \cdot 0.7} = 0.22; \]
\[ cr_{14}^3 = \frac{r_{13} \cdot r_{34}}{r_{13} \cdot r_{34} - (1 - r_{13}) \cdot (1 - r_{34})} = \frac{0.4 \cdot 0.3}{0.4 \cdot 0.3 + 0.6 \cdot 0.7} = 0.22; \]

\[ cr_{14} = \frac{cr_{14}^2 + cr_{14}^3}{2} = 0.22. \]

For \( cr_{41} \) we have:

\[ cr_{41}^2 = \frac{r_{42} \cdot r_{21}}{r_{42} \cdot c_{21} + (1 - c_{42}) \cdot (1 - c_{21})} = \frac{0.31 \cdot 0.3}{0.31 \cdot 0.3 + 0.69 \cdot 0.7} = 0.16; \]

\[ cr_{41}^3 = \frac{r_{43} \cdot r_{31}}{r_{43} \cdot r_{31} - (1 - r_{43}) \cdot (1 - r_{31})} = \frac{0.4 \cdot 0.22}{0.4 \cdot 0.22 + 0.6 \cdot 0.78} = 0.16; \]

\[ cr_{41} = \frac{cr_{41}^2 + cr_{41}^3}{2} = 0.16. \]

**Step 3:** Thus, we obtain the following completed asymmetric fuzzy preference relation \( R \):

\[
R = \begin{pmatrix}
0.5 & 0.4 & 0.4 & 0.22 \\
0.3 & 0.5 & 0.5 & 0.3 \\
0.22 & 0.4 & 0.5 & 0.3 \\
0.16 & 0.31 & 0.4 & 0.5
\end{pmatrix}
\]

**Step 4:** The complete reciprocal intuitionistic fuzzy preference relation is:

\[
B = F^{-1}(R) = \begin{pmatrix}
(0.50, 0.50) & (0.40, 0.30) & (0.40, 0.22) & (0.22, 0.16) \\
(0.30, 0.40) & (0.50, 0.50) & (0.50, 0.40) & (0.30, 0.31) \\
(0.22, 0.40) & (0.40, 0.50) & (0.50, 0.50) & (0.30, 0.40) \\
(0.16, 0.22) & (0.31, 0.30) & (0.40, 0.30) & (0.50, 0.50)
\end{pmatrix}
\]

Notice that the completed reciprocal intuitionistic fuzzy preference relation obtained coincides with the one in [28], where there was a typo in \( b_{41} \) \((b_{41} \) instead of the correct one \((0.16, 0.22) \((0.22, 0.19)) instead of the correct one \((0.16, 0.22) \((0.22, 0.16)) \)

For the following additional incomplete reciprocal intuitionistic fuzzy preference relations [28]

\[
B^2 = \begin{pmatrix}
(0.50, 0.50) & (0.40, 0.45) & x & (0.30, 0.40) \\
(0.45, 0.40) & (0.50, 0.50) & (0.45, 0.40) & x \\
x & (0.40, 0.45) & (0.50, 0.50) & (0.40, 0.55) \\
(0.40, 0.30) & x & (0.55, 0.40) & (0.50, 0.50)
\end{pmatrix}
\]

\[
B^3 = \begin{pmatrix}
(0.50, 0.50) & (0.50, 0.40) & x & (0.40, 0.30) \\
(0.40, 0.50) & (0.50, 0.50) & (0.60, 0.30) & (0.50, 0.40) \\
x & (0.30, 0.60) & (0.50, 0.50) & (0.35, 0.40) \\
(0.30, 0.40) & (0.40, 0.50) & (0.40, 0.35) & (0.50, 0.50)
\end{pmatrix}
\]

\[
B^4 = \begin{pmatrix}
(0.50, 0.50) & (0.40, 0.50) & (0.45, 0.40) & x \\
(0.50, 0.40) & (0.50, 0.50) & (0.50, 0.40) & (0.50, 0.30) \\
(0.40, 0.45) & (0.40, 0.50) & (0.50, 0.50) & (0.50, 0.40) \\
x & (0.30, 0.50) & (0.40, 0.50) & (0.50, 0.50)
\end{pmatrix}
\]
The application of the estimation procedure yields the corresponding completed reciprocal intuitionistic fuzzy preference relations:

\[
B^2 = \begin{pmatrix}
(0.50, 0.50) & (0.40, 0.45) & \mathbf{0.35, 0.33} & (0.30, 0.40) \\
(0.45, 0.40) & (0.50, 0.50) & (0.45, 0.40) & \mathbf{0.31, 0.38} \\
\mathbf{0.33, 0.35} & (0.40, 0.45) & (0.50, 0.50) & (0.40, 0.55) \\
(0.40, 0.30) & \mathbf{0.38, 0.31} & (0.55, 0.40) & (0.50, 0.50)
\end{pmatrix}
\]

\[
B^3 = \begin{pmatrix}
(0.50, 0.50) & (0.50, 0.40) & \mathbf{0.45, 0.20} & (0.40, 0.30) \\
(0.40, 0.50) & (0.50, 0.50) & (0.60, 0.30) & (0.50, 0.40) \\
\mathbf{0.20, 0.45} & (0.30, 0.60) & (0.50, 0.50) & (0.35, 0.40) \\
(0.30, 0.40) & (0.40, 0.50) & (0.40, 0.35) & (0.50, 0.50)
\end{pmatrix}
\]

\[
B^4 = \begin{pmatrix}
(0.50, 0.50) & (0.40, 0.50) & (0.45, 0.40) & \mathbf{0.43, 0.30} \\
(0.50, 0.40) & (0.50, 0.50) & (0.50, 0.40) & (0.50, 0.30) \\
(0.40, 0.45) & (0.40, 0.50) & (0.50, 0.50) & (0.50, 0.40) \\
\mathbf{0.30, 0.43} & (0.30, 0.50) & (0.40, 0.50) & (0.50, 0.50)
\end{pmatrix}
\]

6. Confidence/consistency selection approach with incomplete reciprocal intuitionistic fuzzy preference relations

The aim of the selection process of a group decision making model is to choose the best alternatives according to the opinion of the experts. A classical selection process consists of two phases [9]: (1) aggregation and (2) exploitation. The aggregation phase defines a collective fuzzy preference relation, which indicates the global preference between every ordered pair of alternatives, while the exploitation phase transforms the global information about the alternatives into a global ranking of them, from which a selection set of alternatives is derived.

Confidence has been defined in [36] as a person’s belief that a statement represents the best possible response. Frequently, researchers have found that freely interacting groups choose the positions of their most confident members as their group decisions. This phenomenon has been witnessed with groups discussing a mathematical puzzle [17], a recall task [22] and a recognition task [16], concluding that confidence was a significant predictor of influence. Furthermore Guha et al. state in [12] that in any real field decision making situation when experts give their responses to a particular alternative, their confidence level regarding the opinions are very much important. Therefore in this section a measurement of the expert’s degree of confidence on the opinions provided for reciprocal intuitionistic fuzzy preference relations is defined and used to drive the aggregation of the experts’ preferences.

6.1. Expert’s degree of confidence

Given a reciprocal intuitionistic fuzzy preference relation, the hesitancy degrees used to define confidence measures at its three different levels: pair of alternatives, alternatives and relation levels, as follows:

**Definition 11.** Given a reciprocal intuitionistic fuzzy preference relation \( B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle) \), the confidence level associated to the intuitionistic preference value \( b_{ij} \) is measured as

\[ CFL_{ij} = 1 - \tau_{ij} \]

with \( \tau_{ij} \) being the hesitancy degree associated to \( b_{ij} \).

As noted before in Section 2.2, \( \tau_{ij} = 1 - \mu_{ij} - \nu_{ij} \) and therefore we have that \( CFL_{ij} = \mu_{ij} + \nu_{ij} \). In other words, when \( CFL_{ij} = 1 \) \((\mu_{ij} + \nu_{ij} = 1)\) then \( \tau_{ij} = 0 \) and there is no hesitation at all. The lower the value of \( CFL_{ij} \), the higher the value of \( \tau_{ij} \) and the more hesitation is present in the intuitionistic value \( b_{ij} \).
Definition 12. Given a reciprocal intuitionistic fuzzy preference relation \( B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle) \), the confidence level associated to the alternative \( x_i \) is defined as

\[
CFL_i = \sum_{j=1}^{n} \frac{(CFL_{ij} + CFL_{ji})}{2(n-1)}.
\]

Because \( B \) is reciprocal, we have that \( CFL_{ij} = CFL_{ji} \) (\( \forall i, j \)) and therefore it is

\[
CFL_i = \sum_{j=1}^{n} \frac{CFL_{ij}}{n-1}.
\]

A similar interpretation of \( CFL_i \) with respect to the confidence on the preference values on the alternative \( x_i \) can be done as it was done above with \( CFL_{ij} \).

Definition 13. The confidence level associated to a reciprocal intuitionistic fuzzy preference relation \( B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle) \) is measured as

\[
CFL_B = \frac{\sum_{i=1}^{n} CFL_i}{n}.
\]

Notice that when \( CFL_B = 1 \), then the reciprocal intuitionistic fuzzy preference relation \( B \) is a reciprocal fuzzy preference relation.

6.2. Confidence-consistency guided aggregation

Given a group of experts, their collective preference is obtained by fusing their individual preferences using an appropriate aggregation operator. A widely used aggregation operator in decision making with fuzzy preferences is Yager’s Ordered Weighted Averaging (OWA) operator [32], or one of its extended versions such as the Induced OWA (IOWA) [33].

Definition 14. An IOWA operator of dimension \( m \) is a function \( \Phi_W : (\mathbb{R} \times \mathbb{R})^m \rightarrow \mathbb{R} \), to which a set of weights or weighting vector is associated, \( W = (w_1, \ldots, w_m) \), such that \( w_i \in [0, 1] \) and \( \Sigma_i w_i = 1 \), is expressed as follows:

\[
\Phi_W ((u_1, p_1), \ldots, (u_m, p_m)) = \sum_{i=1}^{m} w_i \cdot p_{\sigma(i)},
\]

being \( \sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\} \) a permutation such that \( u_{\sigma(i)} \geq u_{\sigma(i+1)} \), \( \forall i = 1, \ldots, m-1 \).

Consistency based IOWA operators have been proposed in literature so that the reordering of arguments to aggregate and the computation of the aggregation weights are obtained using consistency degrees values derived from the preferences experts provide [14]. In the case of reciprocal intuitionistic fuzzy preference relation a multiplicative consistency IOWA (MC-IOWA) operator was presented in [28]. These aggregation operators associate higher degree of importance the higher the consistent the preferences are. However, they do not take into consideration the confidence associated to preferences to aggregate, which is proposed to do here by developing a new consistency and confidence IOWA (CC-IOWA) operator, i.e. an IOWA operator that trades off consistency and confidence criteria in both re-ordering the preferences to aggregate and deriving the aggregation weights to use in their fusing to derive the collective preference.
Definition 15 (CC-IOWA operator). Let a set of experts, \( E = \{e_1, \ldots, e_m\} \), provide preferences about a set of alternatives, \( X = \{x_1, \ldots, x_n\} \), using the reciprocal intuitionistic fuzzy preference relations, \( \{B^1, \ldots, B^m\} \). A consistency and confidence IOWA (CC-IOWA) operator of dimension \( m \), \( \Phi_W^{CC} \), is an IOWA operator whose set of order inducing values is the set of consistency/confidence index values, \( \{CCI^1, \ldots, CCI^m\} \), associated with the set of experts.

Therefore, the collective reciprocal intuitionistic fuzzy preference relation \( B^{cc} = (b^{cc}_{ij}) = ((\mu^{cc}_{ij}, \nu^{cc}_{ij})) \) is computed as follows:

\[
\mu^{cc}_{ij} = \Phi_W^{CC} (\langle CCI^1, \mu_{ij}^1 \rangle, \cdots, \langle CCI^m, \mu_{ij}^m \rangle) = \sum_{h=1}^{m} w_h \cdot \mu_{ij}^{\sigma(h)} \tag{5}
\]

\[
\nu^{cc}_{ij} = \Phi_W^{CC} (\langle CCI^1, \nu_{ij}^1 \rangle, \cdots, \langle CCI^m, \nu_{ij}^m \rangle) = \sum_{h=1}^{m} w_h \cdot \nu_{ij}^{\sigma(h)} \tag{6}
\]

\[
CCI^h = (1 - \delta) \cdot CL_h + \delta \cdot CFL_h \tag{7}
\]

such that \( CCI^{\sigma(h-1)} \geq CCI^{\sigma(h)} \), \( w_{\sigma(h-1)} \geq w_{\sigma(h)} \geq 0 \) (\( \forall h \in \{2, \cdots, m\} \)) with \( \sum_{h=1}^{m} w_h = 1 \), \( CL_{ij} \) the consistency level associated to \( B^h = F(B^h) \), \( CFL_{ij} \) the confidence level associated to \( B^h \), and \( \delta \in [0, 1] \) a parameter to control the weight of both consistency and confidence criteria in the inducing variable.

The general procedure for the inclusion of importance weight values, \( \{u_1, \ldots, u_m\} \), in the aggregation process involves the transformation of the values to aggregate under the importance degree to generate a new value and then aggregate these new values using an aggregation operator. In the area of quantifier guided aggregations, Yager provided a procedure to evaluate the overall satisfaction of \( m \) important criteria (experts) by an alternative \( x \) by computing the weighting vector associated to an OWA operator as follows [34]:

\[
w_h = Q \left( \frac{S(h)}{S(m)} \right) - Q \left( \frac{S(h-1)}{S(m)} \right)
\]

being \( Q \) the membership function of the linguistic quantifier, \( S(h) = \sum_{k=1}^{h} u_{\sigma(k)} \), and \( \sigma \) the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with zero importance degree. The linguistic quantifier is a Basic Unit-interval Monotone (BUM) function \( Q : [0, 1] \to [0, 1] \) such that \( Q(0) = 0, Q(1) = 1 \) and if \( x > y \) then \( Q(x) \geq Q(y) \).

Yager extended this procedure to the case of IOWA operator. In this case, each component in the aggregation consists of a triple, with first element being the argument value to aggregate, the second element the importance weight value associated to the first element and the third element being the order inducing value [33]. The same expression as above is used with \( \sigma \) being the permutation that order the induce values from largest to lowest. In our case, we propose to use the consistency/confidence values associated with each expert both as an importance weight and as the order inducing values. Thus, the ordering of the preference values is first induced by the ordering of the experts from the most to the least consistent/confident, and the weights of the CC-IOWA operator is obtained as follows:

\[
w_h = Q \left( \frac{\sum_{k=1}^{h} CCI^{\sigma(k)}}{T} \right) - Q \left( \frac{\sum_{k=1}^{h-1} CCI^{\sigma(k)}}{T} \right)
\]

with \( T = \sum_{k=1}^{m} CCI^k \).
Example 2 (Continuation of Example 1). The first step to compute the collective preference relation is to get the individual consistency and confidence levels.

**Consistency level computation.** For each matrix $R^i$ the matrix $MR^i$ is obtained by applying the estimation procedure using equation (3) and the similarity values between the elements of $R^i$ and $MR^i$ as described in Section 3. To illustrate this procedure we show here only the computation of the estimate of the preference value of the pair of alternatives $(1, 2)$ of $R^1$, $(mr_{12})^1$, and the corresponding consistency index of such pair of alternatives, $(CL_{12})^1$.

The expression of value $(mr_{12})^1$ is

$$ (mr_{12})^1 = \sum_{k \in R_{12}^{01}} \frac{(mr_{k})^1}{\#R_{12}^{01}}; $$

where $R_{12}^{01} = \{ k \neq i, j | (r_{ik})^1, (r_{kj})^1 \notin R^{01} \}$, $R^{01} = \{ (1, 0), (0, 1) \}$, $\#R_{12}^{01}$ is the cardinality of $R_{12}^{01}$ and

$$ (mr_{k})^1 = \frac{(r_{1k})^1 \cdot (r_{k2})^1 \cdot (r_{21})^1}{(r_{2k})^1 \cdot (r_{k1})^1} $$

The values of $(mr_{12})^3$ and $(mr_{12})^4$ are:

$$ (mr_{12})^3 = \frac{(r_{13})^1 \cdot (r_{32})^1 \cdot (r_{21})^1}{(r_{23})^1 \cdot (r_{31})^1} = 0.4 \cdot 0.4 \cdot 0.3 = 0.43636 $$

$$ (mr_{12})^4 = \frac{(r_{14})^1 \cdot (r_{42})^1 \cdot (r_{21})^1}{(r_{24})^1 \cdot (r_{41})^1} = 0.22 \cdot 0.31 \cdot 0.3 = 0.42625 $$

Thus, with two decimal places we have:

$$ (mr_{12})^1 = \frac{(mr_{12})^3 + (mr_{12})^4}{2} = 0.43 $$

and

$$ (CL_{12})^1 = 1 - |0.4 - 0.43| = 0.97 $$

The consistency levels of each individual expert are:

$$ CL^1 = 0.99, CL^2 = 0.99, CL^3 = 0.92, CL^4 = 0.98 $$

**Confidence level computation.** For each reciprocal intuitionistic fuzzy preference relation, $B^i$, its confidence level, $CFL^i$ as described in Section 6.1 is computed, resulting in the following values:

$$ CFL^1 = 0.65, CFL^2 = 0.79, CFL^3 = 0.80, CFL^4 = 0.85 $$
Aggregation. The completed reciprocal intuitionistic fuzzy preference relations are fuses into a collective preference relation by means of the CC-IOWA defined in expressions (5) and (6) using the experts’ consistency-confidence levels CCI as the order inducing variable. To that aim we calculate each expert’s confidence-consistency level following expression (7) with a value of $\delta = 0.5$

$$CCIC_1 = 0.82, CCIC_2 = 0.89, CCIC_3 = 0.86, CCIC_4 = 0.91$$

In order to generate the weighting vector we use the linguistic quantifier “most of” using $Q(r) = r^{1/2}$ [6], and with $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 3$, $\sigma(4) = 1$ the following weights are obtained:

$$w_1 = 0.13, \quad w_2 = 0.21, \quad w_3 = 0.15, \quad w_4 = 0.51.$$  

The collective preference relation $B^{cc}$ is:

$$B^{cc} = \begin{pmatrix} 
(0.50, 0.50) & (0.42, 0.45) & (0.42, 0.33) & (0.37, 0.33) \\
(0.45, 0.42) & (0.50, 0.50) & (0.51, 0.38) & (0.44, 0.33) \\
(0.33, 0.42) & (0.38, 0.51) & (0.50, 0.50) & (0.43, 0.43) \\
(0.30, 0.37) & (0.33, 0.44) & (0.43, 0.43) & (0.50, 0.50) 
\end{pmatrix}$$

6.3. Exploitation phase

At this point, in order to select the alternative(s) ‘best’ acceptable for the majority of the experts and taking advantage of the equivalence between reciprocal intuitionistic fuzzy preference relations and asymmetric fuzzy preference relations we propose the following two quantifier-guided choice degrees of alternatives for the collective reciprocal intuitionistic fuzzy preference relations [13].

1. The Intuitionistic Fuzzy Quantifier Guided Dominance Degree (IFQGDD) for the alternative $x_i$ quantifies the dominance that alternative $x_i$ has over the fuzzy majority of the remaining alternatives:

$$IFQGDD_i = \phi_Q(r_{ij}^{cc}, j = 1, \ldots, n),$$

with $r_{ij}^{cc} = f(b_{ij}^{cc})$ and $\phi_Q$ is an OWA operator guided by the linguistic quantifier represented by the BUM function $Q$.

2. The Intuitionistic Fuzzy Quantifier Guided Non Dominance Degree (IFQGNDD) for the alternative $x_i$ quantifies the degree up to which such alternative is not dominated by a fuzzy majority of the remaining alternatives:

$$IFQGNDD_i = \phi_Q(1 - r_{ji}^*, j = 1, \ldots, n),$$

with $r_{ji}^* = \max \{r_{ji}^{cc} - r_{ij}^{cc}, 0\}$ representing the degree up to which $x_i$ is strictly dominated by $x_j$.

Example 3 (End of Example 1). Using the same linguistic quantifier “most of”, the resultant weighting vector $W_{exp} = (w_{exp1}, w_{exp2}, w_{exp3})$ and quantifier guided dominance and non-dominance degrees are:

$$w_{exp1} = Q(1/3) - Q(0) = 0.58 - 0 = 0.58.$$  
$$w_{exp2} = Q(2/3) - Q(1/3) = 0.82 - 0.58 = 0.24.$$  
$$w_{exp3} = Q(1) - Q(2/3) = 1 - 0.82 = 0.18.$$
7. Conclusions

Uncertainty, hesitation and fuzziness is inherent to all the human being decisions. Therefore in GDM situations it might well be the case of the experts not being able to provide an accurate degree of preference. In these situations reciprocal intuitionistic fuzzy preference relations play a key role since they are able to represent both uncertainty and hesitation, which can be seen as one of the reasons many researchers have turned their research effort to develop theoretical framework for using them in decision making context under uncertainty, of which this paper contributes towards.

The most significant findings and advantages of this contribution are listed below:

• Firstly, we have proved the mathematical equivalence between the set of asymmetric fuzzy preference relations and the set of reciprocal intuitionistic fuzzy preference relations, which can be used to transpose concepts defined in for one preference structure to the other one.

• Indeed, in this paper incomplete reciprocal intuitionistic fuzzy preference relations has been addressed by completing the equivalent incomplete asymmetric fuzzy preference relations using a well known estimation process developed for fuzzy preference relations.

• The concept of confidence level associated to a reciprocal intuitionistic fuzzy preference relation has been defined to associate different importance degrees to experts in the aggregation of individual reciprocal intuitionistic fuzzy preference relations in decision making to derive the collective reciprocal intuitionistic fuzzy preference relation. This concept has been used in conjunction with the consistency level to propose a new consistency and confidence induced ordered weighted averaging (CC-IOWA) operator, in order to implement both consistency and confidence in the resolution process of a group/multicriteria decision making problem.

This contribution opens the door to the development of new methodologies for group decision making that will be addressed in future contributions:

• Development of a consensus approach in which the experts’ confidence level will be taken into account to provide recommendations to increase the consensus as it could be providing recommendations coming from those experts with higher confidence levels.

• Development of a new methodology based on confidence and proximity to estimate the missing preferences in cases where a particular expert is not able to provide any preference for one or more alternatives, a situation that is being described as of total ignorance [2].

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