A family of similarity measures for q-rung orthopair fuzzy sets and their applications to multiple criteria decision making

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Abstract. One worthwhile way of expressing imprecise information is the q-rung orthopair fuzzy sets (q-ROFSs), which extend intuitionistic fuzzy sets and Pythagorean fuzzy sets. The main goal of this contribution is to further extend the concept of similarity measure for q-ROFSs, which not only endows the similarity framework with more ability to create new ones but also inherits all essential properties of a logical similarity measure. This contribution proposes a class of novel similarity measures for q-ROFSs by drawing a general framework of existing q-ROFS similarity and q-ROFS distance measures. These q-ROFS similarity measures enable us to overcome the theoretical drawbacks of the existing measures in the case where they are used individually. In the application part of the contribution, a pattern recognition problem on classification of building materials with a number of known building materials is re-considered. The study of this particular case shows that the proposed family of similarity measures consistently classify the unknown building material pattern with the same known building material pattern. Then, an experimental case study regarding a problem of classroom teaching quality is re-examined for the comparison of the performance of proposed similarity measures against the existing ones. The salient features of the proposed similarity measures in comparison to the existing qROFS similarity measures, are as follows: (i) A number of existing q-ROFS similarity measures are inherently correlation coefficients, and they satisfy only a limited number of essential properties of a comprehensive similarity measure; (ii) Several existing q-ROFS similarity measures lead sometimes to non-logical results, more specifically, to the same maximum similarity value for different q-ROFSs; (iii) A variety of existing q-ROFS similarity measures depend on subjective parameters, which either hinder their application in practice or increase their computational cost. In brief, following this direction of research, we will prove the superiority of the developed

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similarity measures over the existing ones from both theoretical and experimental viewpoints.

**Keywords**: q-rung orthopair fuzzy sets (q-ROFSs), Similarity measure, Multiple criteria decision making.

1 Introduction

Similarity of information is a key concept in decision making processes [1, 3], [5]-[9], [13], [19]-[22], [29, 30, 35, 39], specially, those that are based on consensus, clustering formation, image processing, pattern recognition, machine learning, scheme selection. Indeed, different types of similarity measures have recently appeared in literature, such as the statistical similarity measure introduced to quantify the similarity between two random vectors [11]; the similarity measure used in subtractive clustering algorithm for endmember extraction [36]; the dice similarity measure presented for face recognition problem [28]; and the similarity measure proposed to integrate the traditional fuzzy c-means clustering algorithm [27].

Similarity measure has great practical potential in fuzzy sets [40] and their various extensions, specially those this contribution is concerned with:

1. Intuitionistic fuzzy sets (IFSs) [2] where both the degree of membership and non-membership are considered subject to the constraint of their sum not being above 1;

2. Pythagorean fuzzy sets (PFSs) [37] in which both the degree of membership and non-membership are considered subject to the constraint of their squares addition not being above 1;

3. q-rung orthopair fuzzy sets (q-ROFSs) [38] in which both the degree of membership and non-membership are considered subject to the constraint of their q-th powers addition not being above 1.

The concept of degree of similarity between IFSs was first proposed by Li and Cheng in [13], and a number of similarity measures were put forward which was complemented with discussions on their applications to pattern recognition problems. Then, Mitchell in [22] showed that the maximum similarity value of 1 as modelled by Li and Chen’s similarity measure between IFSs is required to be modified by introducing a stronger condition to avoid the case of different patterns, when it is represented with IFSs, having maximum similarity between them. In the framework of decision making, Xia and Xu [35] inspired by the concept of decision maker’s risk preference introduced a number of parametric similarity measures between IFSs, which the own authors recognised as difficult to be implemented due to the subjectivity in the selection of parameter values. Ye in [39] proposed a cosine similarity measure of IFSs which was based on the mathematical representation of IFSs as vectors in Euclidean space. By considering the cosine similarity measure of IFSs proposed by Ye [39], Tian [30] developed a cotangent-based similarity measure for IFSs with an aim to apply it to medical diagnosis.
While any IFS is a PFS, there are PFSs that are not IFSs. Thus, PFSs when compared with IFSs provide decision makers with an extended set of membership/non-membership degrees for modelling their preferences. As with IFSs, there has been naturally many similarity measures between PFSs proposed in the specialised literature. Zhang [41] proposed a type of normalised Hamming distance based similarity measure between PFSs, which was subsequently used to derive decision making’s priority weights to be used in their preferences aggregation in multiple criteria group decision making. Wei and Wei in [33] introduced a simple extension of Ye’s cosine similarity measure of IFSs [39] and Tian’s cotangent-based similarity measure for IFSs [30] with the inclusion of the hesitation degree and the square powers as per the definition of PFSs. This was also the proposed methodology followed by Rani et al. in [26] in their PFS-based TOPSIS approach.

Recently, Yager introduced the concept of q-ROFSs to represent uncertain information [38]. Yager’s aims were twofold: (1) to provide a general theoretical framework to represent imprecise information while, at the same time, (2) give ”users more freedom in expressing their belief about membership grade”. Indeed, Yager’s q-ROFS extends both PFS and IFS.

One of the leading topics of recent developments in the literature of similarity measures is the attempt to propose a logical similarity measure for q-ROFSs. Wang et al. [34] extended the cosine and cotangent similarity measures for PFSs to q-ROFSs by replacing the square powers with q-powers. Peng and Liu [24] defined ten similarity measures between q-ROFSs as the additive inverse of a same number of distance functions between q-ROFSs. Liu et al. [14] used the average of the cosine and Euclidean based similarity measures for their q-ROFS-based TOPSIS approach, while Peg and Dai [23] developed, following Xia and Xu’s methodology [35], a parametric distance-based similarity measure between q-ROFS.

Although q-ROFS similarity measures have been studied from different viewpoints, the following drawbacks have hampered their application:

- Some existing q-ROFS similarity measures are in fact correlation coefficients, and therefore, they satisfy only a limited number of the essential properties of a comprehensive similarity measure (see e.g. Wang et al. [34]).

- Some existing q-ROFS similarity measures result in the same maximum similarity value for different q-ROFSs (see e.g. Peng and Liu [24]).

- Some existing q-ROFS similarity measures have been constructed as the mean value of q-ROFS distance measure and correlation coefficient and therefore the above first drawback still applies to them (Liu et al. [14]), or depend on subjective parameters which both hinder their application in practice and increase their computational cost (Peg and Dai [23]).
This paper aims to propose a constructive method to design a variety of q-ROFS similarity measures that satisfy a comprehensive set of properties, discriminate efficiently between different q-ROFSs, and do not rely on subjective parameters to decrease the computational cost.

The following section includes the definition of q-ROFS, the established operational laws on the set of q-ROFSs, and the mathematical formulations of the three existing q-ROFS similarity measures as per the above analysis. Section 3 presents an innovative class of q-ROFS similarity measures, which overcome the listed drawbacks of the existing q-ROFS similarity measures. Section 4 presents an application of the proposed q-ROFS similarity measures in MCDM problems to show their superiority over the three existing q-ROFS similarity measures. Section 5 concludes the paper.

2 Preliminaries on q-ROFS similarity measures

It is mentioned in the introduction section that q-ROFS extends both PFS and IFS. In fact, fuzzy set, IFS, PFS and q-ROFS can be seen as types of the same concept of set with different membership/non-membership constraints. The classical set concept is based on total or absence of membership, which is translated to membership/non-membership being in the set \{0, 1\} subject to the constraint non-membership = 1 - membership, i.e. membership + non-membership = 1. The relaxation of the domain of the membership/non-membership only, from the discrete set \{0, 1\} to the continuum set [0, 1], leads to the concept of fuzzy set. When the constraint is jointly relaxed from = to \(\leq\), leads to the concept of IFS with hesitancy defined as the difference between 1 and the addition of membership and non-membership. If in addition both membership and non-membership are modified with a q-power before the constraint is applied, then we have the q-ROFS. When \(q = 1\) a q-ROFS becomes an IFS while with \(q = 2\) a q-ROFS becomes a PFS. This last property is then one used by Yager to claim that q-ROFSs ”greatly increase the modellers ability to capture their judgement of the appropriate orthopair membership grade in given situation” [38].

Formally, given a universal set \(X\), a set \(A\) on \(X\) is characterised by two functions on \(X\) that measure the degree of membership \((\mu_A(x))\) and the degree of non-membership \((\nu_A(x))\) to \(A\) of each element of the universal set \(x \in X\).

- \(A\) is a classical set (CS) when \(\mu_A\) and \(\nu_A\) range over the set \(\{0, 1\}\) and verify the property

\[\mu_A(x) + \nu_A(x) = 1, \quad \forall x \in X.\] (1)

In this case, the membership function is known as characteristic function and it is denoted by \(\delta_A\); the non-membership function is uniquely defined by (1).
- A is a fuzzy set (FS) when $\mu_A$ and $\nu_A$ range over the interval $[0,1]$ and verify the property (1). Consequently, as with CSs, the non-membership function $\nu_A$ is uniquely defined from the membership function.

- A is an intuitionistic fuzzy set (IFS) when $\mu_A$ and $\nu_A$ range over the interval $[0,1]$ and verify the property

$$\mu_A(x) + \nu_A(x) \leq 1, \; \forall x \in X. \quad (2)$$

In this case, given a membership function, there maybe more than one possible non-membership function verifying (2). The concept of hesitancy is therefore present in an IFS, which is modelled in this framework via the hesitancy function $\pi_A = 1 - (\mu_A(x) + \nu_A(x))$.

- A is a Pythagorean fuzzy set (PFS) when $\mu_A$ and $\nu_A$ range over the interval $[0,1]$ and verify the property

$$\mu_A^2(x) + \nu_A^2(x) \leq 1, \; \forall x \in X. \quad (3)$$

As with IFSs, a membership function may have associated more than one non-membership functions as per (3).

- A is a q-rung orthopair fuzzy set (q-ROFS) ($q \geq 1$) when $\mu_A$ and $\nu_A$ range over the interval $[0,1]$ and verify the property

$$\mu_A^q(x) + \nu_A^q(x) \leq 1, \; \forall x \in X. \quad (4)$$

If $q = 2$, then (4) becomes (3); while if $q = 1$, then (4) becomes (2).

Given an element $x \in X$, the notation $(\mu_A(x), \nu_A(x))$ is referred to as a q-rung orthopair fuzzy number (q-ROFN), and it is simply denoted as $A = (\mu_A, \nu_A)$.

Remark 2.1 In the next discussions, we denote both ROFS $A$ and ROFN $A$ with the same symbol $A$, letting the context tell which is meant.

Below, the relevant set and algebraical operations on q-ROFNs are listed [38]:

Given any q-ROFNs $A$ and $B$, we can obtain:

Complement : $A^c = (\mu_A^c, \nu_A^c) = (\nu_A, \mu_A)$; \hfill (5)

Intersection : $A \cap B = (\mu_{A \cap B}, \nu_{A \cap B}) = (\min\{\mu_A, \mu_B\}, \max\{\nu_A, \nu_B\});$ \hfill (6)

Union : $A \cup B = (\mu_{A \cup B}, \nu_{A \cup B}) = (\max\{\mu_A, \mu_B\}, \min\{\nu_A, \nu_B\});$ \hfill (7)

Inclusion : $A \subseteq B \; \text{if and only if} \; \mu_A \leq \mu_B \; \text{and} \; \nu_A \geq \nu_B;$ \hfill (8)
2.1 Distance and similarity measures between q-ROFNs

The intention here is to give the axiomatic skeleton of comprehensive and acceptable q-ROFS distance and similarity measures. Remember that for the universal set $X$, $q$–ROFS($X$) denotes the set of q-ROFSs on $X$ while $q$–ROFN($X$) denotes the set of all possible q-ROFNs available to allocate to elements of $X$.

In what follows, we will assume that the universal set $X$ is of finite cardinality, and therefore, any q-ROFS on $X$ will have finite number of q-ROFNs.

Definition 2.2 [6] A distance measure between q-ROFNs is a mapping $D : q$–ROFN($X$) × $q$–ROFN($X$) → $[0, 1]$ which satisfies the properties:

(D1) $D(A, B) = D(B, A)$;
(D2) $D(A, B) = 0$ if and only if $A = B$;
(D3) $D(A, A^c) = 1$ if and only if $A$ is in $\{(1, 0), (0, 1)\}$;
(D4) If $A \subseteq B \subseteq C$, then $D(A, C) \geq D(A, B)$ and $D(A, C) \geq D(B, C)$.

Definition 2.3 [6] A similarity measure between q-ROFNs is a mapping $S : q$–ROFN($X$) × $q$–ROFN($X$) → $[0, 1]$ which satisfies the properties:

(S1) $S(A, B) = S(B, A)$;
(S2) $S(A, B) = 1$ if and only if $A = B$;
(S3) $S(A, A^c) = 0$ if and only if $A$ is in $\{(1, 0), (0, 1)\}$;
(S4) If $A \subseteq B \subseteq C$, then $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$.

2.2 Existing similarity measures for q-ROFSs

Below are listed the mathematical formulation of the main existing similarity measures on q-ROFSs proposed in the specialised literature.

Peng and Liu [24]’s similarity measures for q-ROFSs
Peng and Liu [24]'s q-ROFS similarity measures are q-ROFS distance-based as per the transform

\[ S(A, B) = 1 - d(A, B). \]

They proposed the below 10 similarity measures on q-ROFS:

\[
S_{PL1}(A, B) = 1 - \frac{1}{2|X|} \sum_{x \in X} (|\mu_A^p(x) - \mu_B^p(x)| + |\nu_A^p(x) - \nu_B^p(x)| + |\pi_A^p(x) - \pi_B^p(x)|); \tag{13}
\]
\[
S_{PL2}(A, B) = 1 - \frac{1}{2|X|} \sum_{x \in X} (|\mu_A^p(x) - \mu_B^p(x)| - (\nu_A^p(x) - \nu_B^p(x)))); \tag{14}
\]
\[
S_{PL3}(A, B) = 1 - \frac{1}{4|X|} (\sum_{x \in X} (|\mu_A^p(x) - \mu_B^p(x)| + |\nu_A^p(x) - \nu_B^p(x)| + |\pi_A^p(x) - \pi_B^p(x)|)
+ \sum_{x \in X} (|\mu_A^p(x) - \nu_A^p(x)| - (\mu_B^p(x) - \nu_B^p(x)))); \tag{15}
\]
\[
S_{PL4}(A, B) = 1 - \frac{1}{|X|} \sum_{x \in X} \max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|\}; \tag{16}
\]
\[
S_{PL5}(A, B) = \frac{1}{|X|} \sum_{x \in X} \frac{1 - \max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|\}}{1 + \max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|\}}; \tag{17}
\]
\[
S_{PL6}(A, B) = \frac{\sum_{x \in X} (1 - \max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|\})}{\sum_{x \in X} (1 + \max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|\})}; \tag{18}
\]
\[
S_{PL7}(A, B) = \lambda_1 \sum_{x \in X} \min\{|\mu_A^p(x), \mu_B^p(x)|, \mu_A^p(x), \mu_B^p(x)|\} + \lambda_2 \sum_{x \in X} \max\{|\nu_A^p(x), \nu_B^p(x)|, \nu_A^p(x), \nu_B^p(x)|\}; \tag{19}
\]
\[
S_{PL8}(A, B) = \lambda_1 \sum_{x \in X} \min\{|\mu_A^p(x), \mu_B^p(x)|, \max\{|\nu_A^p(x), \nu_B^p(x)|\} + \lambda_2 \sum_{x \in X} \max\{|\nu_A^p(x), \nu_B^p(x)|, \max\{|\nu_A^p(x), \nu_B^p(x)|\}; \tag{20}
\]
\[
S_{PL9}(A, B) = \frac{1}{|X|} \sum_{x \in X} \min\{|\mu_A^p(x), \mu_B^p(x)|, \mu_A^p(x), \mu_B^p(x)| + \min\{|\nu_A^p(x), \nu_B^p(x)|, \nu_A^p(x), \nu_B^p(x)|\}; \tag{21}
\]
\[
S_{PL10}(A, B) = \frac{\sum_{x \in X} (\min\{|\mu_A^p(x), \mu_B^p(x)|, \min\{|\nu_A^p(x), \nu_B^p(x)|, \nu_A^p(x), \nu_B^p(x)|\})}{\sum_{x \in X} (\max\{|\mu_A^p(x), \mu_B^p(x)|, \max\{|\nu_A^p(x), \nu_B^p(x)|, \nu_A^p(x), \nu_B^p(x)|\}); \tag{22}
\]

Wang et al. [34]'s similarity measures for q-ROFS

Wang et al. [34] proposed the following q-ROFS similarity measures:

- q-ROF weighted cosine-based similarity measures:

\[
S_{W1}(A, B) = \sum_{x \in X} \omega_x \frac{\mu_A^p(x)\mu_B^p(x) + \nu_A^p(x)\nu_B^p(x)}{\sqrt{\mu_A^p(x) + \nu_A^p(x) + \mu_B^p(x) + \nu_B^p(x)}}; \tag{23}
\]
\[
S_{W2}(A, B) = \sum_{x \in X} \omega_x \frac{\mu_A^p(x)\mu_B^p(x) + \nu_A^p(x)\nu_B^p(x) + \pi_A^p(x)\pi_B^p(x)}{\sqrt{\mu_A^p(x) + \nu_A^p(x) + \pi_A^p(x) + \mu_B^p(x) + \nu_B^p(x) + \pi_B^p(x)}}; \tag{24}
\]

- q-ROF cosine similarity based measures:

\[
S_{W3}(A, B) = \sum_{x \in X} \omega_x \cos\frac{1}{2} (\max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|\}); \tag{25}
\]
\[
S_{W4}(A, B) = \sum_{x \in X} \omega_x \cos\frac{1}{2} (\max\{|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|, |\pi_A^p(x) - \pi_B^p(x)|\}); \tag{26}
\]
\[ S_{W5}(A, B) = \sum_{x \in X} \omega_x \cos\left[\frac{\pi}{4} (|\mu_A^q(x) - \mu_B^q(x)|, |\nu_A^q(x) - \nu_B^q(x)|)\right]; \quad (27) \]
\[ S_{W6}(A, B) = \sum_{x \in X} \omega_x \cos\left[\frac{\pi}{4} (|\mu_A^q(x) - \mu_B^q(x)|, |\nu_A^q(x) - \nu_B^q(x)|, |\pi_A^q(x) - \pi_B^q(x)|)\right]; \quad (28) \]

- q-ROF cotangent similarity based measures:

\[ S_{WT}(A, B) = \sum_{x \in X} \omega_x \cot\left[\frac{\pi}{4} + \frac{\pi}{4} (\max\{|\mu_A^q(x) - \mu_B^q(x)|, |\nu_A^q(x) - \nu_B^q(x)|\})\right]; \quad (29) \]
\[ S_{WS}(A, B) = \sum_{x \in X} \omega_x \cos\left[\frac{\pi}{4} + \frac{\pi}{4} (\max\{|\mu_A^q(x) - \mu_B^q(x)|, |\nu_A^q(x) - \nu_B^q(x)|\})\right]; \quad (30) \]
\[ S_{W9}(A, B) = \sum_{x \in X} \omega_x \cos\left[\frac{\pi}{4} + \frac{\pi}{8} (|\mu_A^p(x) - \mu_B^p(x)|, |\nu_A^p(x) - \nu_B^p(x)|)\right]; \quad (31) \]
\[ S_{W10}(A, B) = \sum_{x \in X} \omega_x \cos\left[\frac{\pi}{4} + \frac{\pi}{8} (|\mu_A^q(x) - \mu_B^q(x)|, |\nu_A^q(x) - \nu_B^q(x)|, |\pi_A^q(x) - \pi_B^q(x)|)\right]; \quad (32) \]

where each \( \omega_x \in [0, 1] \) stands for the weight of element \( x \in X \) subject to \( \sum_{x \in X} \omega_x = 1 \).

**Peng and Dai [23]’s similarity measure for q-ROFSs**

Peng and Dai [23] proposed the following q-ROF distance based similarity measure:

\[ S_{PD}(A, B) = 1 - \left(\frac{1}{2t_k^q} \sum_{j=1}^{n} \omega_j \right)(t_k - 1)(\mu_A^p(x) - \mu_B^p(x) - (\nu_A^p(x) - \nu_B^p(x)))^q \]
\[ + |(t_k - k)(\nu_A^p(x) - \nu_B^p(x)) - k(\mu_A^p(x) - \mu_B^p(x))|^q \right]^\frac{1}{q}; \quad (33) \]

where \( q \) stands for the LP norm, \( p \) denotes the power, \( k \) and \( t_k \) are respectively the slope and the level of uncertainty related to \( k \) such that \( t_k \geq k + 1 \) for \( k \geq 0 \). Moreover, \( \omega_j \in [0, 1] \) represents the weighting value of element \( x_i \in X \) subject to \( \sum_{j=1}^{n} \omega_j = 1 \).

### 3 New similarity measures for q-ROFSs

This section proposes a class of novel similarity measures for q-ROFSs by drawing a general framework of existing q-ROFS similarity and q-ROFS distance measures.

Suppose that \( S(A, B) \) and \( D(A, B) \) are respectively q-ROFS similarity and q-ROFS distance measures, but not necessarily the reciprocal ones. Then, we define

\[ S_{\Pi \Pi}(A, B) = \prod \left[ S(A, B), \prod [D(A, B)] \right], \quad (34) \]

where the operator \( \prod \) is strictly monotone increasing with respect to each argument. In particular, the below operators are proposed:

\[ \prod_t [r, t] = \frac{r + t}{2}; \quad (35) \]
\[ \Pi_2[r,t] = \frac{(r + t)^2}{4}; \quad (36) \]
\[ \Pi_3[r,t] = \frac{2(r + t)}{2 + r + t}; \quad (37) \]
\[ \Pi_4[r,t] = \frac{r + t}{2} e^{\frac{r + t}{2} - 1}, \quad (38) \]

and the operator \( \Pi \) is strictly monotone decreasing with respect to its argument. In particular, the below operators are proposed:

\[ \Pi_1[t] = 1 - t; \quad (39) \]
\[ \Pi_2[t] = \frac{1 - t}{1 + \alpha t}, \quad \alpha > -1; \quad (40) \]
\[ \Pi_3[t] = (1 - t^\beta)^{\frac{1}{\beta}}, \quad \beta > 0, \quad (41) \]

In the case where \( D(A,B) \) stands for a q-ROFS distance measure, then, we will be able to construct a q-ROFS similarity measure directly by the use of the following transforms (see [4]):

\[ S(A,B) = 1 - D(A,B); \quad (42) \]
\[ S(A,B) = \frac{1}{1 + D(A,B)}; \quad (43) \]
\[ S(A,B) = 1 - D^2(A,B); \quad (44) \]
\[ S(A,B) = \cos(D(A,B)); \quad (45) \]
\[ S(A,B) = e^{-D(A,B)}. \quad (46) \]

By the way, in the above proposed rule given by (34), there is no obligation to be concerned about the direct relationship between q-ROFS similarity measure \( S \) and q-ROFS distance measure \( D \). This means that we enable the researcher to construct a wide range of similarity measures \( S_{\Pi_1} \) for q-ROFSs by choosing not necessarily reciprocal similarity and distance measures as that given by (42)-(46). In fact, such a methodology helps researchers to produce a wide set of similarity measures from one side, and from the other side, the hybrid similarity measures as the combination of different existing similarity and distance measures offsets the drawbacks of individual ones.

Using the above definition of \( S_{\Pi_1} \), we are now able to construct a variety of q-ROFS similarity measures as follows:

\[ S_{11}(A,B) = \frac{S(A,B) + 1 - D(A,B)}{2}; \]
\[ S_{12}(A,B) = \frac{S(A,B) + \frac{1 - D(A,B)}{1 + D(A,B)}}{2}, \quad \alpha > -1; \]
\[ S_{13}(A,B) = \frac{S(A,B) + (1 - D^\beta(A,B))^{\frac{1}{\beta}}}{2}, \quad \beta > 0; \]
\[ S_{21}(A, B) = \frac{|S(A, B) + 1 - D(A, B)|^2}{4}; \]
\[ S_{22}(A, B) = \frac{|S(A, B) + 1 - D(A, B)|^2}{4}, \quad \alpha > -1; \]
\[ S_{23}(A, B) = \frac{|S(A, B) + (1 - D^\beta(A, B))^\frac{1}{\beta}|^2}{4}; \quad \beta > 0; \]
\[ S_{31}(A, B) = \frac{\frac{2S(A, B) + 1 - D(A, B)}{2 + S(A, B) + 1 - D(A, B)}}{\frac{1 + \alpha D(A, B)}{2 + S(A, B) + 1 - D(A, B)}}, \quad \alpha > -1; \]
\[ S_{32}(A, B) = \frac{\frac{2S(A, B) + 1 - D^\beta(A, B)}{2 + S(A, B) + 1 - D^\beta(A, B)}}{\frac{1 + \alpha D(A, B)}{2 + S(A, B) + 1 - D^\beta(A, B)}}, \quad \beta > 0; \]
\[ S_{41}(A, B) = \frac{\frac{S(A, B) + 1 - D(A, B)}{2} \times e^{\frac{S(A, B) + 1 - D(A, B)}{2} - 1}}{\frac{1 + \alpha D(A, B)}{2 + S(A, B) + 1 - D(A, B)}}, \quad \alpha > -1; \]
\[ S_{42}(A, B) = \frac{\frac{S(A, B) + 1 - D(A, B)}{2} \times e^{\frac{S(A, B) + 1 - D(A, B)}{2} - 1}}{\frac{1 + \alpha D(A, B)}{2 + S(A, B) + 1 - D(A, B)}}, \quad \beta > 0; \]

If we set \( S := S_{PL}(A, B) \) and \( D := D_{W3}(A, B) \) in the above formulas, then the following \( q \)-ROFS similarity measures are obtained:
\[
S_{11}(A, B) = \frac{SP_{L1}(A, B) + 1 - D_{W3}(A, B)}{2}
\]

\[
S_{12}(A, B) = \frac{SP_{L2}(A, B) + 1 - D_{W3}(A, B)}{2}
\]

\[
S_{21}(A, B) = \frac{SP_{L1}(A, B) + 1 - D_{W3}(A, B)}{2}
\]

\[
S_{22}(A, B) = \frac{SP_{L2}(A, B) + 1 - D_{W3}(A, B)}{2}
\]

\[
S_{31}(A, B) = \frac{SP_{L3}(A, B) + 1 - D_{W3}(A, B)}{2}
\]

\[
S_{32}(A, B) = \frac{SP_{L2}(A, B) + 1 - D_{W3}(A, B)}{2}
\]

\[
S_{33}(A, B) = \frac{SP_{L3}(A, B) + 1 - D_{W3}(A, B)}{2}
\]
\[
0 < \theta' < \frac{1}{4} (g V)^{-1} \leq \left( \frac{(g V)^{-1} + (g V')^1 + \alpha_g}{(g V)^{-1} + (g V')^1 + \alpha_g} \right) \leq \left( \frac{(g V)^{-1} + (g V')^1 + \alpha_g}{(g V)^{-1} + (g V')^1 + \alpha_g} \right)^2
\]
However, the defined q-ROFS similarity measure $S_{\prod\bigvee}$ enables us to overcome the aforementioned drawbacks of the existing measures when used individually.

**Theorem 3.1** $S_{\prod\bigvee}$ given by (34) is a q-ROFS similarity measure.

**Proof.** The proof of (S1) and (S2) is clear.

[Proof of (S3)]: Let $S_{\prod\bigvee}(A, B) = 1$. Then, with respect to the definition (34), we have

$$\prod [S(A, B), \prod [D(A, B)]] = 1.$$ 

Case ($\prod = \prod_{i=1}^{n}$, $\bigvee = \bigvee_{j=1,2,3}$): The above relation results that $S(A, B) + \prod_j [D(A, B)] = 1$ for any $j = 1, 2, 3$.

The latter relation holds true if and only if

$$S(A, B) = 1, \text{ and } \prod_j [D(A, B)] = 1, \text{ for any } j = 1, 2, 3,$$

if and only if

$$S(A, B) = 1, \text{ and } D(A, B) = 0.$$ 

It immediately follows from the properties (S3) for $S$ together with (D3) for $D$ that $A = B$.

[Proof of (S4)]: Let $S_{\prod\bigvee}(A, B) = 0$. Then, with respect to the definition (34), we have

$$\prod [S(A, B), \prod [D(A, B)]] = 0.$$ 

Case ($\prod = \prod_{i=1}^{n}$, $\bigvee = \bigvee_{j=1,2,3}$): The above relation gives rise to $S(A, B) + \prod_j [D(A, B)] = 0$ for any $j = 1, 2, 3$.

The latter relation holds true if and only if

$$S(A, B) = 0, \text{ and } \prod_j [D(A, B)] = 0, \text{ for any } j = 1, 2, 3,$$

if and only if

$$S(A, B) = 0, \text{ and } D(A, B) = 1.$$ 

(70)

It immediately follows from the properties (S4) for $S$ together with (D4) for $D$ that (70) holds true if and only if $B = A^c$.

[Proof of (S5)]: Suppose that $A \subseteq B \subseteq C$. Then, from the properties (S5) together with (D5) it concludes that $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$ together with $D(A, C) \geq D(A, B)$ and $D(A, C) \geq D(B, C)$.

Now, as follows from strictly monotone increasing property of $\prod_i$, and strictly monotone decreasing property of $\bigvee_j$, we can get

$$\prod_i [S(A, B), \prod_j [D(A, B)]] \leq \prod_i [S(A, C), \prod_j [D(A, C)]]$$.
and

\[ \prod_i [S(B, C), \prod_j [D(B, C)]] \leq \prod_i [S(A, C), \prod_j [D(A, C)]], \]

and therefore,

\[ S_{\prod_j}(A, B) \leq S_{\prod_j}(A, C), \]

and

\[ S_{\prod_j}(B, C) \leq S_{\prod_j}(A, C). \]

The other cases \( \prod_1 = \prod_{i=2,3,4}, \prod_2 = \prod_{j=1,2,3} \) can be proved similarly, and therefore, they are not considered here. □

4 Applications of the proposed q-ROFS similarity measures

In this part of the contribution, decision making problems analyzed in [34] and [23] are re-consider to illustrate the feasibility of the proposed q-ROFS similarity measures. Experimental results show the effectiveness of the proposed similarity measures in comparison with the existing q-ROFS similarity measures.

4.1 Case study I

The pattern recognition problem used in Wang et al. [34] is used to investigate the applicability of the proposed q-ROFS similarity measures and compare their performance with respect to Wang et al.’s similarity measures.

Weng et al. considered a pattern recognition problem about classification of building materials with five known building materials and below information on them, which are described as q-ROFSs on the reference set \( X = \{x_1, x_2, x_3, x_4, x_5\} \):

\[
A_1 = \{(x_1, 0.5, 0.8), (x_2, 0.6, 0.4), (x_3, 0.8, 0.3), (x_4, 0.6, 0.9), (x_5, 0.1, 0.4)\};
\]

\[
A_2 = \{(x_1, 0.6, 0.7), (x_2, 0.7, 0.3), (x_3, 0.6, 0.2), (x_4, 0.8, 0.6), (x_5, 0.3, 0.5)\};
\]

\[
A_3 = \{(x_1, 0.3, 0.4), (x_2, 0.7, 0.5), (x_3, 0.9, 0.3), (x_4, 0.4, 0.8), (x_5, 0.2, 0.3)\};
\]

\[
A_4 = \{(x_1, 0.5, 0.3), (x_2, 0.4, 0.4), (x_3, 0.6, 0.2), (x_4, 0.4, 0.7), (x_5, 0.2, 0.6)\};
\]

\[
A_5 = \{(x_1, 0.4, 0.7), (x_2, 0.2, 0.6), (x_3, 0.5, 0.4), (x_4, 0.5, 0.3), (x_5, 0.4, 0.2)\}.
\]

In this problem, we are going to classify the below unknown building material pattern

\[
A = \{(x_1, 0.7, 0.6), (x_2, 0.8, 0.2), (x_3, 0.4, 0.3), (x_4, 0.7, 0.8), (x_5, 0.4, 0.2)\},
\]
with one of the classes $A_i$ $(i = 1, 2, 3, 4, 5)$.

Following Wang et al. [34] and assuming that $p = 3$, we employ the proposed family of 12 similarity measures given by (47)-(69) in this contribution. Table 1 shows the results obtained using the 10 similarity measures proposed by Wang et al. [34] and the 12 similarity measures obtained with the proposed approach.

Results using the weighted similarity measures proposed by Wang et al. [34] with their proposed weighting vector $w = (0.15, 0.20, 0.25, 0.10, 0.30)$ is shown in Table 2, which also includes the results using Liu and Wang [16]’s distance-based q-ROFWA operator and q-ROFGA operator.
Similarity measures | $S(A_1,*)$ | $S(A_2,*)$ | $S(A_3,*)$ | $S(A_4,*)$ | $S(A_5,*)$
--- | --- | --- | --- | --- | ---
Wang et al. [34]'s measures
$S_{W1}$ | 0.7433 | **0.8003** | 0.7988 | 0.7345 | 0.6897
$S_{W2}$ | 0.8795 | 0.9116 | **0.9124** | 0.8766 | 0.8543
$S_{W3}$ | 0.8975 | **0.9588** | 0.8496 | 0.9057 | 0.8654
$S_{W4}$ | 0.9559 | **0.9774** | 0.9498 | 0.9561 | 0.9291
$S_{W5}$ | 0.8975 | **0.9588** | 0.8364 | 0.8880 | 0.8540
$S_{W6}$ | 0.8964 | **0.9630** | 0.8386 | 0.8701 | 0.8362
$S_{W7}$ | 0.6618 | **0.7633** | 0.6362 | 0.6613 | 0.6766
$S_{W8}$ | 0.7571 | **0.8257** | 0.7613 | 0.7544 | 0.7522
$S_{W9}$ | 0.6618 | **0.7633** | 0.6198 | 0.6318 | 0.6596
$S_{W10}$ | 0.6588 | **0.7702** | 0.6259 | 0.6085 | 0.6496
Proposed measures
$S_{11}$ | 0.3940 | **0.4904** | 0.4188 | 0.4147 | 0.3940
$S_{12}$ | 0.3745 | **0.4566** | 0.3853 | 0.3824 | 0.3745
$S_{13}$ | 0.3541 | **0.4199** | 0.3489 | 0.3476 | 0.3541
$S_{21}$ | 0.1552 | **0.2404** | 0.1754 | 0.1719 | 0.1552
$S_{22}$ | 0.1402 | **0.2085** | 0.1484 | 0.1463 | 0.1402
$S_{23}$ | 0.1254 | **0.1764** | 0.1217 | 0.1208 | 0.1254
$S_{31}$ | 0.5653 | **0.6580** | 0.5904 | 0.5862 | 0.5653
$S_{32}$ | 0.5449 | **0.6269** | 0.5563 | 0.5533 | 0.5449
$S_{33}$ | 0.5230 | **0.5915** | 0.5173 | 0.5159 | 0.5230
$S_{41}$ | 0.2149 | **0.2946** | 0.2342 | 0.2309 | 0.2149
$S_{42}$ | 0.2004 | **0.2652** | 0.2084 | 0.2062 | 0.2004
$S_{43}$ | 0.1856 | **0.2351** | 0.1819 | 0.1810 | 0.1856

Table 1. Similarity values between classes $A_i$ ($i = 1, 2, 3, 4, 5$) and the pattern $* \equiv A_i$ ($p = 3$).
Similarity measures

**Wang et al. [34]’s measures**

<table>
<thead>
<tr>
<th></th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(S_4)</th>
<th>(S_5)</th>
</tr>
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<td>0.7553</td>
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<tr>
<td>3</td>
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<td>0.8398</td>
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<tr>
<td>4</td>
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<td>0.9464</td>
</tr>
<tr>
<td>5</td>
<td>0.8962</td>
<td><strong>0.9673</strong></td>
<td>0.8299</td>
<td>0.8986</td>
<td>0.8910</td>
</tr>
<tr>
<td>6</td>
<td>0.8961</td>
<td><strong>0.9693</strong></td>
<td>0.8303</td>
<td>0.8883</td>
<td>0.8830</td>
</tr>
<tr>
<td>7</td>
<td>0.6740</td>
<td><strong>0.7831</strong></td>
<td>0.6478</td>
<td>0.6735</td>
<td>0.7474</td>
</tr>
<tr>
<td>8</td>
<td>0.7740</td>
<td><strong>0.8482</strong></td>
<td>0.7700</td>
<td>0.7733</td>
<td>0.8065</td>
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<tr>
<td>9</td>
<td>0.6740</td>
<td><strong>0.7831</strong></td>
<td>0.6356</td>
<td>0.6522</td>
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<tr>
<td>10</td>
<td>0.6727</td>
<td><strong>0.7866</strong></td>
<td>0.6356</td>
<td>0.6389</td>
<td>0.7284</td>
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</tbody>
</table>

**Distance-based q-ROFWA operator**
(Liu and Wang [16])

<table>
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<tr>
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<th>(S_3)</th>
<th>(S_4)</th>
<th>(S_5)</th>
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</thead>
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<td>0.0275</td>
<td>0.0222</td>
<td>0.0197</td>
</tr>
</tbody>
</table>

**Distance-based q-ROFWG operator**
(Liu and Wang [16])

Proposed measures

<table>
<thead>
<tr>
<th></th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(S_4)</th>
<th>(S_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8368</td>
<td><strong>0.9074</strong></td>
<td>0.7679</td>
<td>0.8143</td>
<td>0.8368</td>
</tr>
<tr>
<td>2</td>
<td>0.7951</td>
<td><strong>0.8921</strong></td>
<td>0.7100</td>
<td>0.7772</td>
<td>0.7951</td>
</tr>
<tr>
<td>3</td>
<td>0.6192</td>
<td><strong>0.7592</strong></td>
<td>0.5279</td>
<td>0.6053</td>
<td>0.6192</td>
</tr>
<tr>
<td>4</td>
<td>0.7003</td>
<td><strong>0.8234</strong></td>
<td>0.5897</td>
<td>0.6632</td>
<td>0.7003</td>
</tr>
<tr>
<td>5</td>
<td>0.6322</td>
<td><strong>0.7958</strong></td>
<td>0.5041</td>
<td>0.6041</td>
<td>0.6322</td>
</tr>
<tr>
<td>6</td>
<td>0.3834</td>
<td><strong>0.5764</strong></td>
<td>0.2787</td>
<td>0.3663</td>
<td>0.3834</td>
</tr>
<tr>
<td>7</td>
<td>0.9112</td>
<td><strong>0.9515</strong></td>
<td>0.8687</td>
<td>0.8977</td>
<td>0.9112</td>
</tr>
<tr>
<td>8</td>
<td>0.8859</td>
<td><strong>0.9430</strong></td>
<td>0.8304</td>
<td>0.8747</td>
<td>0.8859</td>
</tr>
<tr>
<td>9</td>
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<td><strong>0.8631</strong></td>
<td>0.6910</td>
<td>0.7541</td>
<td>0.7648</td>
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<tr>
<td>10</td>
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<td><strong>0.8272</strong></td>
<td>0.6089</td>
<td>0.6764</td>
<td>0.7108</td>
</tr>
<tr>
<td>11</td>
<td>0.6478</td>
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<td>0.5312</td>
<td>0.6220</td>
<td>0.6478</td>
</tr>
<tr>
<td>12</td>
<td>0.4231</td>
<td><strong>0.5968</strong></td>
<td>0.3292</td>
<td>0.4079</td>
<td>0.4231</td>
</tr>
</tbody>
</table>

**Table 2.** Weighted similarity values between classes \(A_i\) \((i = 1, 2, 3, 4, 5)\) and the pattern \(* \sim A.\) \((p = 3).\)

Tables 1 and 2 clearly shows that the proposed family of similarity measures consistently classify the unknown building material pattern with the same known building material pattern, which is not the case with Wang et al.’s similarity measures with or without weights.
4.2 Case study II

Following Peng and Dai [23]'s work, we consider here a set of alternatives \( \{A_1, A_2, \ldots, A_m\} \) with respect to a set of criteria \( \{C_1, C_2, \ldots, C_n\} \) with weighting vector \((w_1, w_2, \ldots, w_n)\) such that \(0 \leq w_j \leq 1\) and \(\sum_{j=1}^{n} w_j = 1\).

The q-ROF matrix \( D \) whose elements representing the evaluation of \( A_i \) \((i = 1, 2, \ldots, m)\) with respect to \( C_j \) \((j = 1, 2, \ldots, n)\) is built

\[
D = [D^{(ij)}]_{m \times n} = \begin{pmatrix}
(\mu_{D(11)}, \nu_{D(11)}) & (\mu_{D(12)}, \nu_{D(12)}) & \cdots & (\mu_{D(1n)}, \nu_{D(1n)}) \\
(\mu_{D(21)}, \nu_{D(21)}) & (\mu_{D(22)}, \nu_{D(22)}) & \cdots & (\mu_{D(2n)}, \nu_{D(2n)}) \\
\vdots & \vdots & \ddots & \vdots \\
(\mu_{D(m1)}, \nu_{D(m1)}) & (\mu_{D(m2)}, \nu_{D(m2)}) & \cdots & (\mu_{D(mn)}, \nu_{D(mn)})
\end{pmatrix}.
\]

Like in Peng and Dai [23], elements of the q-ROF matrix are normalised

\[
(\tilde{\mu}_{D^{(ij)}}, \tilde{\nu}_{D^{(ij)}}) = \begin{cases}
(\mu_{D^{(ij)}}, \nu_{D^{(ij)}}), & \text{if } C_j \text{ is benefit;} \\
(\nu_{D^{(ij)}}, \mu_{D^{(ij)}}), & \text{if } C_j \text{ is cost},
\end{cases}
\]

to guarantee criteria compatibility.

Using \( A^* = \{1, 0\} \) as the ideal alternative, Peng and Dai [23] proposed the following q-ROFS similarity measure-based approach:

**Algorithm 4.1 (A q-ROFS similarity measure-based algorithm)**

**Step 1.** Input the q-ROF decision matrix \( D = [D^{(ij)}]_{m \times n}; \)

**Step 2.** Transform the q-ROF decision matrix \( D = [D^{(ij)}]_{m \times n} \) into the normalized matrix \( \tilde{D} = [\tilde{D}^{(ij)}]_{m \times n} \) by using the rule of (71);

**Step 3.** Compute the q-ROFS similarity scores \( S(A_i = (\mu_i, \nu_i), A^*) \) for any \( i = 1, 2, \ldots, m; \)

**Step 4.** Rank alternatives using the decreasing ranking of their corresponding similarity scores.

Note that any of the q-ROFS similarity measures proposed in this contribution (47)-(69) may be used in Step 3 above.

On the basis of Algorithm 4.1 one can choose the desired alternative by calculating the similarity measure between each alternative \( A_i \) and the ideal alternative \( A^* \). Intuitively, the greater the value of \( S(A_i, A^*) \), the better the alternative; and the lower the value of \( S(A_i, A^*) \), the less important the alternative. Thus, the similarity measure plays an important role in ranking alternatives classified by the decision makers.

In the below, a classroom teaching quality, first studied by Zhang et al. [42] and thoroughly re-examined by Peng and Dai [23], is used to compare the proposed similarity measures against Peng and Dai’s similarity measure.
Five teachers \( A_i \) (\( i = 1, 2, 3, 4, 5 \)) are invited to assess the classroom teaching quality with respect to five benefit type criteria \( C_1 \): teaching attitude, \( C_2 \): teaching capacity, \( C_3 \): teaching content, \( C_4 \): teaching method, and \( C_5 \): teaching effect. It is also assumed the following experts’ weighting vector: \( w = (\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = (0.3, 0.2, 0.14, 0.16, 0.2) \). The q-ROF decision matrix is given in Table 3.

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>(0.85,0.15)</td>
<td>(0.85,0.15)</td>
<td>(0.55,0.15)</td>
<td>(0.65,0.25)</td>
<td>(0.45,0.15)</td>
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<tr>
<td>( A_2 )</td>
<td>(0.65,0.2)</td>
<td>(0.85,0.2)</td>
<td>(0.45,0.2)</td>
<td>(0.55,0.35)</td>
<td>(0.35,0.15)</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>(0.45,0.15)</td>
<td>(0.75,0.25)</td>
<td>(0.6,0.35)</td>
<td>(0.85,0.45)</td>
<td>(0.55,0.25)</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>(0.65,0.15)</td>
<td>(0.7,0.2)</td>
<td>(0.6,0.15)</td>
<td>(0.75,0.35)</td>
<td>(0.35,0.15)</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>(0.7,0.25)</td>
<td>(0.85,0.25)</td>
<td>(0.55,0.3)</td>
<td>(0.65,0.25)</td>
<td>(0.45,0.2)</td>
</tr>
</tbody>
</table>

**Table 3.** The qROF matrix with \( p = 3 \).

Table 4 shows that the ROF-CODAS algorithm [10], the q-ROFS similarity measure algorithm [23] and the proposed algorithm in this contribution all result in the same optimal solution.
Algorithm of ROF-CODAS [10]  
qu-ROFS similarity measure algorithm [23]


\[
\begin{array}{ccc}
\text{Algorithm 4.1 based on} & \text{Ranking} & \text{Optimal alternative} \\
& & \\
\text{the proposed similarity measure} & A_1 > A_5 > A_4 > A_3 > A_2 & A_1 \\
& & \\
S_{11} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.7250 0.3081 0.3315 0.3210 0.3793 \\
S_{12} & A_1 > A_5 > A_4 > A_3 > A_2 & A_1 \\
& & 0.6451 0.2373 0.2570 0.2475 0.2995 \\
S_{13} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.5334 0.1464 0.1585 0.1512 0.1877 \\
S_{21} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.5256 0.0950 0.1099 0.1030 0.1439 \\
S_{22} & A_1 > A_5 > A_4 > A_3 > A_2 & A_1 \\
& & 0.4162 0.0563 0.0660 0.0612 0.0897 \\
S_{23} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.2845 0.0214 0.0251 0.0229 0.0352 \\
S_{31} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.8405 0.4711 0.4979 0.4860 0.5500 \\
S_{32} & A_1 > A_5 > A_4 > A_3 > A_2 & A_1 \\
& & 0.7843 0.3836 0.4089 0.3968 0.4609 \\
S_{33} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.6957 0.2555 0.2736 0.2627 0.3161 \\
S_{41} & A_1 > A_5 > A_4 > A_3 > A_2 & A_1 \\
& & 0.5506 0.1543 0.1699 0.1628 0.2039 \\
S_{42} & A_1 > A_5 > A_3 > A_4 > A_2 & A_1 \\
& & 0.4524 0.1107 0.1222 0.1166 0.1486 \\
S_{43} & A_1 > A_5 > A_4 > A_3 > A_2 & A_1 \\
& & 0.3345 0.0624 0.0683 0.0647 0.0833 \\
\end{array}
\]

Table 4. The ranking orders of teachers $A_i$ ($i = 1, 2, 3, 4, 5$) and the optimum one. $p = 3$, $q = 3$, $t_k = 3$, $k = 1$.

In what follows, we illustrate further two more optimization problems which were previously analysed by Peng and Dai [23] using some aggregation-based decision-making techniques. There, they assumed that the assessments for teachers are in the form of q-ROF matrices given in Tables 5 and 7. The corresponding results
together with the results of the proposed algorithm are given in Tables 6 and 8.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.85,0.15)</td>
<td>(0.85,0.15)</td>
<td>(0.75,0.15)</td>
<td>(0.75,0.25)</td>
<td>(0.85,0.15)</td>
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<tr>
<td>$A_2$</td>
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<td>(0.55,0.2)</td>
<td>(0.65,0.35)</td>
<td>(0.75,0.15)</td>
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<td>$A_3$</td>
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<td>(0.5,0.35)</td>
<td>(0.65,0.45)</td>
<td>(0.75,0.25)</td>
</tr>
<tr>
<td>$A_4$</td>
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<td>(0.8,0.2)</td>
<td>(0.5,0.15)</td>
<td>(0.55,0.45)</td>
<td>(0.65,0.15)</td>
</tr>
<tr>
<td>$A_5$</td>
<td>(0.13,0.12)</td>
<td>(1.0)</td>
<td>(1.0)</td>
<td>(0.11,0.09)</td>
<td>(0.15,0.12)</td>
</tr>
</tbody>
</table>

**Table 5.** The q-ROF matrix with $p = 3$. 
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Ranking</th>
<th>Optimal alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm of CODAS [23]</td>
<td>$A_1 &gt; A_5 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>q-ROFS similarity measure algorithm [23]</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_4 &gt; A_5$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>q-ROFWA-based algorithm [16]</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td>q-ROFWG-based algorithm [16]</td>
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</tr>
<tr>
<td>q-ROFWBM-based algorithm [15]</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td>q-ROFWGBM-based algorithm [15]</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_4 &gt; A_5$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>q-ROFWGHM-based algorithm [32]</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_4 &gt; A_5$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>q-ROFWHM-based algorithm [17]</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td>Algorithm 4.1 based on the proposed similarity measure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_5 &gt; A_4$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$S_{12}$</td>
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<td>$A_1$</td>
</tr>
<tr>
<td>$S_{13}$</td>
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<td>$A_1$</td>
</tr>
<tr>
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<td>$S_{22}$</td>
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<td>$S_{23}$</td>
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<td>$S_{31}$</td>
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<tr>
<td>$S_{32}$</td>
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<td>$A_1$</td>
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<tr>
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<tr>
<td>$S_{41}$</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_5 &gt; A_4$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$S_{42}$</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_5 &gt; A_4$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$S_{43}$</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_5 &gt; A_4$</td>
<td>$A_1$</td>
</tr>
</tbody>
</table>

Table entries are in the form of similarity measures calculated for different algorithms and alternatives.
Table 6. The ranking order of teachers $A_i$ ($i = 1, 2, 3, 4, 5$) and the optimum one corresponding to the data in Table 5. $p = 3$, $q = 3$ and $t_k = 3$, $k = 1$.

From Table 6, we easily observe that the most appropriate alternative is $A_1$. However, as discussed before, Peng and Dai’s [23] combinative distance-based assessment (CODAS) and similarity measure give rise to the same maximum similarity value for different q-ROFSs, and all the aggregation-based algorithms including q-ROFWA-based and q-ROFWG-based algorithms [16], q-ROFWBM-based and q-ROFWGBM-based algorithms [15], q-ROFWGHM-based algorithm [32] and q-ROFHWBM-based algorithm [17] have higher computation complexity than Peng and Dai’s [23] algorithms. This is while, the proposed algorithms do not address these disadvantages and provide more accuracy decision-making results.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.65,0.25)</td>
<td>(0.25,0.05)</td>
<td>(0.35,0.25)</td>
<td>(0.35,0.15)</td>
<td>(0.23,0.12)</td>
</tr>
<tr>
<td>$A_2$</td>
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<td>(0.25,0.25)</td>
<td>(0.35,0.25)</td>
<td>(0.25,0.15)</td>
<td>(0.21,0.11)</td>
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<tr>
<td>$A_3$</td>
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<td>(0.15,0.05)</td>
<td>(0.25,0.15)</td>
<td>(0.25,0.05)</td>
<td>(0.21,0.13)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.35,0.15)</td>
<td>(0.15,0.05)</td>
<td>(0.15,0.15)</td>
<td>(0.15,0.05)</td>
<td>(0.15,0.12)</td>
</tr>
<tr>
<td>$A_5$</td>
<td>(0.99,0.11)</td>
<td>(0,1)</td>
<td>(0,1)</td>
<td>(0.99,0.01)</td>
<td>(0.99,0.01)</td>
</tr>
</tbody>
</table>

Table 7. The q-ROF matrix with $p = 3$.  

23
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Ranking</th>
<th>Optimal alternative</th>
</tr>
</thead>
<tbody>
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<td>Algorithm of ROF-CODAS [10]</td>
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</tr>
<tr>
<td>q-ROFWA-based algorithm [16]</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td>q-ROFWG-based algorithm [16]</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_4 &gt; A_5$</td>
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<tr>
<td>q-ROFWBM-based algorithm [15]</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
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<td>$A_1$</td>
</tr>
<tr>
<td>q-ROFWGHM-based algorithm [32]</td>
<td>$A_1 &gt; A_2 &gt; A_3 &gt; A_4 &gt; A_5$</td>
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</tr>
<tr>
<td>q-ROFWHM-based algorithm [17]</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
</tbody>
</table>

Algorithm 4.1 based on the proposed similarity measure

<table>
<thead>
<tr>
<th>$S_{11}$</th>
<th>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</th>
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</tr>
</thead>
<tbody>
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<tr>
<td>$S_{12}$</td>
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<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0922 0.0576 0.0312 0.0137 0.5061</td>
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</tr>
<tr>
<td>$S_{13}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0536 0.0330 0.0177 0.0077 0.4068</td>
<td></td>
</tr>
<tr>
<td>$S_{21}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0163 0.0066 0.0020 0.0004 0.4223</td>
<td></td>
</tr>
<tr>
<td>$S_{22}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0005 0.0330 0.0010 0.0002 0.3204</td>
<td></td>
</tr>
<tr>
<td>$S_{23}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0029 0.0011 0.0003 0.0001 0.1655</td>
<td></td>
</tr>
<tr>
<td>$S_{51}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.2263 0.1499 0.0849 0.0384 0.7878</td>
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</tr>
<tr>
<td>$S_{32}$</td>
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<td>$A_5$</td>
</tr>
<tr>
<td></td>
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<td></td>
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<tr>
<td>$S_{33}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.1018 0.0638 0.0347 0.0153 0.5784</td>
<td></td>
</tr>
<tr>
<td>$S_{41}$</td>
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<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0533 0.0323 0.0171 0.0073 0.4579</td>
<td></td>
</tr>
<tr>
<td>$S_{42}$</td>
<td>$A_5 &gt; A_1 &gt; A_2 &gt; A_3 &gt; A_4$</td>
<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0372 0.0225 0.0118 0.0051 0.3668</td>
<td></td>
</tr>
<tr>
<td>$S_{43}$</td>
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<td>$A_5$</td>
</tr>
<tr>
<td></td>
<td>0.0208 0.0125 0.0066 0.0028 0.2248</td>
<td></td>
</tr>
</tbody>
</table>
Table 8. The ranking order of teachers $A_i$ ($i = 1, 2, 3, 4, 5$) and the optimum one corresponding to the data in Table 7. $p = 3$, $q = 3$, $t_k = 3$, $k = 1$.

Again, from Table 8, we easily find that the most appropriate alternative is $A_5$, and as indicated by Peng and Dai [23], $A_1$ is the unreasonable result due to the counter-intuitive phenomena. This verifies that the proposed algorithms perform more effectively than the existing ones.

5 Conclusions and further research perspectives

The purpose of this article was to develop a new class of similarity measures for q-ROFSs, which provides the interested researcher with a variety of similarity measures that inherit all acceptable and essential properties of a logical similarity measure. The comparison of proposed q-ROFS similarity measures with the existing ones in a pattern recognition problem about classification of building materials with a number of known building materials, and a problem of classroom teaching quality to compare the performance of proposed similarity measures against the existing ones, showed their superiority from both theoretical and experimental viewpoints.

Further, the proposal methodology allows to derive functional conditions to be verified by a q-ROF similarity function, and it addressed a number of drawbacks associated with existing similarity measures of q-ROFS: (1) a number of existing q-ROFS similarity measures satisfy only a limited set of the essential properties needed for a comprehensive similarity measure because of their correlation orientation; (2) a set of existing q-ROFS similarity measures do not make a difference between two actual different q-ROFSs; (3) some existing q-ROFS similarity measures depend on subjective parameters which both hinder their application in practice and increase their computational cost.

Indeed, comparative analysis shows that new family of similarity measures consistently produce the same final best decision in almost all cases, which is not the case with some of the existing similarity measures.

Eventually, future works may be further extended by applying the proposed similarity measures to fields that require reference to

- defining a class of reasonable comparative techniques;
- developing q-ROFS information measures including distance, similarity and entropy measures;
- q-ROFSs based group consensus measures; and
- decision making approaches under q-ROF environment.
Compliance with Ethical Standards

Conflict of Interest

The author declares that he has no conflict of interest.

Human and Animal Rights

This article does not contain any studies with human participants or animals performed by the author.

References


[41] Zhang XL. A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making. *Int J Intell Syst*, 2016;31:593-611.

Figure 1: Decision making algorithm (Algorithm 4.1).

Algorithm 4.1

Normalization process
- Step 1. Input the q-ROF decision matrix $D$
- Step 2. Normalize $D$ into $\overline{D}$

Decision process
- Step 3. Compute the q-ROFS similarity measure $\delta(A_i, A^*)$

Selection Process
- Step 4. Characterize the ranking order of alternatives $A_i$. 