Most existing distances between intuitionistic fuzzy sets are defined in linear plane representations in 2D or 3D space. Here, we define a new interpretation of intuitionistic fuzzy sets as a restricted spherical surface in 3D space. A new spherical distance for intuitionistic fuzzy sets is introduced. We prove that the spherical distance is different from those existing distances in that it is nonlinear with respect to the change of the corresponding fuzzy membership degrees.

1. INTRODUCTION

Research in cognition science\(^1\) has shown that people are faster at identifying an object that is significantly different from other objects than at identifying an object similar to others. The semantic distance between objects plays a significant role in the performance of these comparisons.\(^2\) For the concepts represented by fuzzy sets and intuitionistic fuzzy sets, an element with full membership (nonmembership) is usually much easier to be determined because of its categorical difference from other elements. This requires the distance between intuitionistic fuzzy sets or fuzzy sets to reflect the semantic context of where the membership/nonmembership values are, rather than a simple relative difference between them.

Most existing distances based on the linear representation of intuitionistic fuzzy sets are linear in nature, in the sense of being based on the relative difference between membership degrees.\(^3\)–\(^15\) Obviously, in some semantic contexts these distances might not seem to be the most appropriate ones. In such cases, nonlinear distances between intuitionistic fuzzy sets may be more adequate to capture the semantic difference. Here, new nonlinear distances between two intuitionistic fuzzy sets are introduced. We call these distances spherical distances because their definition is based on a spherical representation of intuitionistic fuzzy sets.

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The paper is set out as follows. In the following section, some preliminary definitions and notation on intuitionistic fuzzy sets needed throughout the paper are provided. In Section 3, a review of the existing geometrical interpretations of intuitionistic fuzzy sets and the distance functions proposed and usually used in the literature is given. The spherical interpretation of intuitionistic fuzzy sets and spherical distance functions are introduced in Section 4. Because fuzzy sets are particular cases of intuitionistic fuzzy sets, the corresponding spherical distance functions for fuzzy sets are derived in Section 5. Finally, Section 6 presents our conclusion.

2. **INTUITIONISTIC FUZZY SETS: PRELIMINARIES**

Intuitionistic fuzzy sets were introduced by Atanassov in Ref. 3. The following provides its definition, which will be needed throughout the paper:

**Definition 1. (Intuitionistic fuzzy set)** An intuitionistic fuzzy set $A$ of the universe of discourse $U$ is given by

$$A = \{ \langle u, \mu_A(u), \nu_A(u) \rangle | u \in U \}$$

where

$$\mu_A : U \to [0, 1], \quad \nu_A : U \to [0, 1]$$

and

$$0 \leq \mu_A(u) + \nu_A(u) \leq 1 \quad \forall u \in U.$$  

For each $u$, the numbers $\mu_A(u)$ and $\nu_A(u)$ are the degree of membership and degree of nonmembership of $u$ to $A$, respectively.

Another concept related to intuitionistic fuzzy sets is the hesitancy degree, $\tau_A(u) = 1 - \mu_A(u) - \nu_A(u)$, which represents the hesitance to the membership of $u$ to $A$.

We note that the main difference between intuitionistic fuzzy sets and traditional fuzzy sets resides in the use of two parameters for membership degrees instead of a single value. Obviously, $\mu_A(u)$ represents the lowest degree of $u$ belonging to $A$, and $\nu_A(u)$ gives the lowest degree of $u$ not belonging to $A$. If we consider $\mu = v^-$ as membership and $\nu = 1 - v^+$ as nonmembership, then we come up with the so-called interval-valued fuzzy sets $[v^-, v^+]$.\(^{16}\)

**Definition 2. (Interval-valued fuzzy set)** An interval-valued fuzzy set $A$ of the universe of discourse $U$ is given by

$$A = \{ \langle u, M_A(u) \rangle | u \in U \}$$
where the function

\[ M_A : U \to \mathcal{P}[0, 1] \]

defines the degree of membership of an element \( u \) to \( A \), being

\[ \mathcal{P}[0, 1] = \{[a, b], \ a, b \in [0, 1], \ a \leq b \} \]

Interval-valued fuzzy sets\(^{16,17}\) and intuitionistic fuzzy sets\(^{3–5}\) emerged from different grounds and thus they have associated different semantics.\(^{18}\) However, they have been proven to be mathematically equivalent.\(^{19–23}\) and because of this we do not distinguish them throughout this paper. For the sake of convenience when comparing existing distances, we will apply the notation of intuitionistic fuzzy sets. Our conclusions could easily be adapted to interval-valued fuzzy sets.

3. EXISTING GEOMETRICAL REPRESENTATIONS OF AND DISTANCES BETWEEN INTUITIONISTIC FUZZY SETS

In contrast to traditional fuzzy sets where only a single number is used to represent membership degree, more parameters are needed for intuitionistic fuzzy sets. Geometrical interpretations have been associated with these parameters,\(^ {24}\) which are especially useful when studying the similarity or distance between sets. One of these geometrical interpretations of intuitionistic fuzzy sets was given by Atanassov in Ref. 5, as shown in Figure 1a, where a universe \( U \) and subset \( OST \) in the Euclidean plane with Cartesian coordinates are represented. According to this interpretation, given an intuitionistic fuzzy set \( A \), a function \( f_A \) from \( U \) to \( OST \) can be constructed such that if \( u \in U \), then

\[ P = f_A(u) \in OST \]

is the point with coordinates \((\mu_A(u), \nu_A(u))\) for which \( 0 \leq \mu_A(u) \leq 1, \ 0 \leq \nu_A(u) \leq 1, \ 0 \leq \mu_A(u) + \nu_A(u) \leq 1 \).

We note that the triangle \( OST \) in Figure 1a is an orthogonal projection of the 3D representation proposed by Szmidt and Kacprzyk,\(^ {10}\) as shown in Figure 1b. In Figure 1b, in addition to \( \mu_A(u) \) and \( \nu_A(u) \), a third dimension is present, \( \tau_A(u) = 1 - \mu_A(u) - \nu_A(u) \). Because \( \mu_A(u) + \nu_A(u) + \tau_A(u) = 1 \), the restricted plane \( RST \) can be interpreted as the 3D counterpart of an intuitionistic fuzzy set. Therefore, in a similar way to Atanassov’s procedure, for an intuitionistic fuzzy set \( A \) a function \( f_A \) from \( U \) to \( RST \) can be constructed in such a way that given \( u \in U \), then

\[ S = f_A(u) \in RST \]

has coordinates \((\mu_A(u), \nu_A(u), \tau_A(u))\) for which \( 0 \leq \mu_A(u) \leq 1, \ 0 \leq \nu_A(u) \leq 1, \ 0 \leq \tau_A(u) \leq 1 \) and \( \mu_A(u) + \nu_A(u) + \tau_A(u) = 1 \).
A fuzzy set is a special case of an intuitionistic fuzzy set where $\tau_A(x) = 0$ holds for all elements. In this case, both $OST$ and $RST$ in Figure 1 converge to the segment $ST$. Therefore, under this interpretation, the distance between two fuzzy sets is based on the membership functions, which depends on just one parameter (membership). Given any two fuzzy subsets $A = \{ (u_i, \mu_A(u_i)) : u_i \in U \}$ and $B = \{ (u_i, \mu_B(u_i)) : u_i \in U \}$ with $U = \{ u_1, u_2, \ldots, u_n \}$, the following distances have been proposed:

- **Hamming distance $d_1(A, B)$**

  $$d_1(A, B) = \sum_{i=1}^{n} |\mu_A(u_i) - \mu_B(u_i)|$$

- **Normalized Hamming distance $l_1(A, B)$**

  $$l_1(A, B) = \frac{1}{n} \sum_{i=1}^{n} |\mu_A(u_i) - \mu_B(u_i)|$$

- **Euclidean distance $e_1(A, B)$**

  $$e_1(A, B) = \sqrt{\sum_{i=1}^{n} (\mu_A(u_i) - \mu_B(u_i))^2}$$

- **Normalized Euclidean distance $q_1(A, B)$**

  $$q_1(A, B) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\mu_A(u_i) - \mu_B(u_i))^2}$$
In the case of intuitionistic fuzzy sets, different distances have been defined according to either of the 2D or 3D interpretations. Atanassov in Ref. 5 presents the following distances for any two intuitionistic fuzzy subsets \( A = \{ (u_i, \mu_A(u_i), \nu_A(u_i)) : u_i \in U \} \) and \( B = \{ (u_i, \mu_B(u_i), \nu_B(u_i)) : u_i \in U \} \) using the above 2D interpretation:

- **Hamming distance** \( d_2(A, B) \)

\[
d_2(A, B) = \frac{1}{2} \sum_{i=1}^{n} [ |\mu_A(u_i) - \mu_B(u_i)| + |\nu_A(u_i) - \nu_B(u_i)| ]
\]

- **Normalized Hamming distance** \( l_2(A, B) \)

\[
l_2(A, B) = \frac{1}{2n} \sum_{i=1}^{n} [ |\mu_A(u_i) - \mu_B(u_i)| + |\nu_A(u_i) - \nu_B(u_i)| ]
\]

- **Euclidean distance** \( e_2(A, B) \)

\[
e_2(A, B) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} (\mu_A(u_i) - \mu_B(u_i))^2 + (\nu_A(u_i) - \nu_B(u_i))^2 }
\]

- **Normalized Euclidean distance** \( q_2(A, B) \)

\[
q_2(A, B) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} (\mu_A(u_i) - \mu_B(u_i))^2 + (\nu_A(u_i) - \nu_B(u_i))^2 }
\]

Based on the 3D representation, Szmidt and Kacprzyk\(^9,10\) modified these distances to include the third parameter \( \tau_A(u) \):

- **Hamming distance** \( d_3(A, B) \)

\[
d_3(A, B) = \frac{1}{2} \sum_{i=1}^{n} [ |\mu_A(u_i) - \mu_B(u_i)| + |\nu_A(u_i) - \nu_B(u_i)| + |\tau_A(u_i) - \tau_B(u_i)| ]
\]

- **Normalized Hamming distance** \( l_3(A, B) \)

\[
l_3(A, B) = \frac{1}{2n} \sum_{i=1}^{n} [ |\mu_A(u_i) - \mu_B(u_i)| + |\nu_A(u_i) - \nu_B(u_i)| + |\tau_A(u_i) - \tau_B(u_i)| ]
\]

- **Euclidean distance** \( e_3(A, B) \)

\[
e_3(A, B) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} (\mu_A(u_i) - \mu_B(u_i))^2 + (\nu_A(u_i) - \nu_B(u_i))^2 + (\tau_A(u_i) - \tau_B(u_i))^2 }
\]
Normalized Euclidean distance $q_3(A, B)$

$$q_3(A, B) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left[ (\mu_A(u_i) - \mu_B(u_i))^2 + (v_A(u_i) - v_B(u_i))^2 + (\tau_A(u_i) - \tau_B(u_i))^2 \right]}$$

All the above distances clearly adopt the linear plane interpretation, as shown in Figure 1, and therefore they reflect only the relative differences between the memberships, nonmemberships, and hesitancy degrees of intuitionistic fuzzy sets. The following lemma illustrates this characteristic:

**Lemma 1.** Let $A = \{\langle u_i, \mu_A(u_i), v_A(u_i) \rangle : u_i \in U \}$, $B = \{\langle u_i, \mu_B(u_i), v_B(u_i) \rangle : u_i \in U \}$, $C = \{\langle u_i, \mu_C(u_i), v_C(u_i) \rangle : u_i \in U \}$, and $G = \{\langle u_i, \mu_G(u_i), v_G(u_i) \rangle : u_i \in U \}$ be four intuitionistic fuzzy sets of the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. If the following conditions hold

$$\mu_A(u_i) - \mu_B(u_i) = \mu_C(u_i) - \mu_G(u_i)$$
$$v_A(u_i) - v_B(u_i) = v_C(u_i) - v_G(u_i)$$

then

$$D(A, B) = D(C, G)$$

being $D$ any of the above Atanassov’s 2D or Szmidt and Kacprzyk’s 3D distance functions.

**Proof.** The proof is obvious for Atanassov’s 2D distances. For Szmidt and Kacprzyk’s 3D distances, the proof follows from the fact that

$$\mu_A(u_i) - \mu_B(u_i) = \mu_C(u_i) - \mu_G(u_i) \land v_A(u_i) - v_B(u_i) = v_C(u_i) - v_G(u_i)$$

imply

$$\tau_A(u_i) - \tau_B(u_i) = \tau_C(u_i) - \tau_G(u_i)$$

If $|\tau_A(u_i) - \tau_B(u_i)| = |\tau_C(u_i) - \tau_G(u_i)|$, then conditions in Lemma 1 can be generalized to

$$|\mu_A(u_i) - \mu_B(u_i)| = |\mu_C(u_i) - \mu_G(u_i)|$$
$$|v_A(u_i) - v_B(u_i)| = |v_C(u_i) - v_G(u_i)|$$
This means that if we move both sets in the space shown in Figure 1b with the same changes in membership, nonmembership, and hesitancy degrees, then we obtain exactly the same distance between the two fuzzy sets. This linear feature of the above distances may not be adequate in some cases, because the human perception is not necessarily always linear. For example, we can classify the human behavior as perfect, good, acceptable, poor, and worst. Using fuzzy sets, we can assign their fuzzy membership as 1, 0.75, 0.5, 0.25, and 0. To find out if someone’s behavior is perfect or not, we only need to check if there is anything wrong with him. However, to differentiate good from acceptable, we have to count their positive and negative points. Obviously, the semantic distance between perfect and good should be greater than the semantic distance between good and acceptable. This semantic difference is not shown by the linear distances between their memberships. Therefore, a nonlinear representation of the distance between two intuitionistic fuzzy sets may benefit the representative power of intuitionistic fuzzy sets. Although nonlinearity could be modeled by using many different expressions, we will consider and use a simple one to model it. Here, we propose a new geometrical interpretation of intuitionistic fuzzy sets in 3D space using a restricted spherical surface. This new representation provides a convenient nonlinear measure of the distance between two intuitionistic fuzzy sets.

4. SPHERICAL INTERPRETATION OF INTUITIONISTIC FUZZY SETS: SPHERICAL DISTANCE

Let $A = \{ (u, \mu_A(u), \nu_A(u)) : u \in U \}$ be an intuitionistic fuzzy set. We have

$$\mu_A(u) + \nu_A(u) + \tau_A(u) = 1$$

which can be equivalently transformed to

$$x^2 + y^2 + z^2 = 1$$

with

$$x^2 = \mu_A(u), \quad y^2 = \nu_A(u), \quad z^2 = \tau_A(u)$$

It is obvious that this is not the only transformation that could be used. However, as shown in the existing distances, there is no special reason to discriminate $\mu_A(u)$, $\nu_A(u)$, and $\tau_A(u)$. Therefore, a simple nonlinear transformation to unit sphere in a 3D Euclidean space is selected here, as shown in Figure 2.

This transformation allows us to interpret an intuitionistic fuzzy set as a restricted spherical surface. An immediate consequence of this interpretation is that the distance between two elements of an intuitionistic fuzzy set can be defined as the spherical distance between their corresponding points on its restricted spherical surface representation. This distance is defined as the shortest path between the two
points, i.e. the length of the arc of the great circle passing through both points. For points \( P \) and \( Q \) in Figure 2, their spherical distance is\(^{25}\).

\[
d_s(P, Q) = \arccos \left\{ 1 - \frac{1}{2} \left[ (x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2 \right] \right\}.
\]

This expression can be used to obtain the spherical distance between two intuitionistic fuzzy sets, \( A = \{ (u_i, \mu_A(u_i), \nu_A(u_i)) : u_i \in U \} \) and \( B = \{ (u_i, \mu_B(u_i), \nu_B(u_i)) : u_i \in U \} \) of the universe of discourse \( U = \{ u_1, u_2, \ldots, u_n \} \), as follows:

\[
d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left\{ 1 - \frac{1}{2} \left[ (\sqrt{\mu_A(u_i)} - \sqrt{\mu_B(u_i)})^2 + (\sqrt{\nu_A(u_i)} - \sqrt{\nu_B(u_i)})^2 \right] \right\}
\]

where the factor \( \frac{2}{\pi} \) is introduced to get distance values in the range \([0, 1]\) instead of \([0, \frac{\pi}{2}]\). Because \( \mu_A(u_i) + \nu_A(u_i) + \tau_A(u_i) = 1 \) and \( \mu_B(u_i) + \nu_B(u_i) + \tau_B(u_i) = 1 \), we have that

\[
d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)\mu_B(u_i)} + \sqrt{\nu_A(u_i)\nu_B(u_i)} + \sqrt{\tau_A(u_i)\tau_B(u_i)} \right)
\]

This is summarized in the following definition:
DEFINITION 3. (Spherical distance) Let \( A = \{ (u_i, \mu_A(u_i), v_A(u_i)) : u_i \in U \} \) and \( B = \{ (u_i, \mu_B(u_i), v_B(u_i)) : u_i \in U \} \) be two intuitionistic fuzzy sets of the universe of discourse \( U = \{ u_1, u_2, \ldots, u_n \} \). The spherical and normalized spherical distances between \( A \) and \( B \) are

\begin{itemize}
  \item Spherical distance \( d_s(A, B) \)
    \[
    d_s(A, B) = \frac{2 \pi}{n} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)\mu_B(u_i)} + \sqrt{v_A(u_i)v_B(u_i)} + \sqrt{\tau_A(u_i)\tau_B(u_i)} \right)
    \]
  \item Normalized spherical distance \( d_{ns}(A, B) \)
    \[
    d_{ns}(A, B) = \frac{2}{n\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)\mu_B(u_i)} + \sqrt{v_A(u_i)v_B(u_i)} + \sqrt{\tau_A(u_i)\tau_B(u_i)} \right)
    \]
\end{itemize}

Clearly, we have that \( 0 \leq d_s(A, B) \leq n \) and \( 0 \leq d_{ns}(A, B) \leq 1 \).

Different from the distances in Section 3, the proposed spherical distances implement in their definition not only the difference between membership, non-membership, and hesitancy degrees but also their actual values. This is shown in the following result:

LEMMA 2. Let \( A = \{ (u_i, \mu_A(u_i), v_A(u_i)) : u_i \in U \} \) and \( B = \{ (u_i, \mu_B(u_i), v_B(u_i)) : u_i \in U \} \) be two intuitionistic fuzzy subsets of the universe of discourse \( U = \{ u_1, u_2, \ldots, u_n \} \). Let \( a = \{ a_1, a_2, \ldots, a_n \} \) and \( b = \{ b_1, b_2, \ldots, b_n \} \) be two sets of real numbers (constants). If the following conditions hold for each \( u_i \in U \):

\[
\mu_B(u_i) = \mu_A(u_i) + a_i \\
v_B(u_i) = v_A(u_i) + b_i
\]

then the following inequalities hold:

\[
\frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( 1 - e_i^2 \right) \leq d_s(A, B) \leq \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( 1 - c_i \right) \left( 1 - |a_i| - |b_i| \right)
\]

\[
\frac{2}{n\pi} \sum_{i=1}^{n} \arccos \left( 1 - e_i^2 \right) \leq d_{ns}(A, B) \leq \frac{2}{n\pi} \sum_{i=1}^{n} \arccos \left( 1 - c_i \right) \left( 1 - |a_i| - |b_i| \right)
\]
where \( c_i = \max\{|a_i|, |b_i|\} \) and \( e_i = \min\{|a_i|, |b_i|\} \). The maximum distance between \( A \) and \( B \) is obtained if and only if one of them is a fuzzy set or their available information supports only opposite membership degree for each one.

**Proof.** See Appendix A.

Owing to the nonlinear characteristic of the spherical distance, they do not satisfy Lemma 1. However, the following properties hold for the spherical distances:

**Lemma 3.** Let \( A = \{\langle u_i, \mu_A(u_i), \nu_A(u_i) \rangle: u_i \in U\}, B = \{\langle u_i, \mu_B(u_i), \nu_B(u_i) \rangle: u_i \in U\} \) and \( E = \{\langle u_i, \mu_E(u_i), \nu_E(u_i) \rangle: u_i \in U\} \) be three intuitionistic fuzzy subsets of the universe of discourse \( U = \{u_1, u_2, \ldots, u_n\} \). Let \( a = \{a_1, a_2, \ldots, a_n\} \) and \( b = \{b_1, b_2, \ldots, b_n\} \) two sets of real positive numbers (constants) satisfying the following conditions:

\[
|\mu_B(u_i) - \mu_A(u_i)| = a_i, \quad |\nu_B(u_i) - \nu_A(u_i)| = b_i
\]

\[
|\mu_E(u_i) - \mu_A(u_i)| = a_i, \quad |\nu_E(u_i) - \nu_A(u_i)| = b_i
\]

If \( E \) is one of the two extreme crisp sets with either \( \mu_E(u_i) = 1 \) or \( \nu_E(u_i) = 1 \) for all \( u_i \in U \), then the following inequalities hold:

\[
d_s(A, B) < d_s(A, E), \quad d_{ns}(A, B) < d_{ns}(A, E)
\]

The distance between intuitionistic fuzzy sets \( A \) and \( B \) is always lower than the distance between \( A \) and the extreme crisp sets \( E \) under the same difference of their memberships and nonmemberships.

**Proof.** We provide the proof just for the extreme fuzzy set with full memberships, being the proof for full nonmembership similar.

With \( E \) being the extreme crisp set with full memberships, we have

\[
\mu_E(u_i) = 1, \quad \nu_E(u_i) = 0, \quad \tau_E(u_i) = 0
\]

Because \( |\mu_E(u_i) - \mu_A(u_i)| = a_i \) and \( |\nu_E(u_i) - \nu_A(u_i)| = b_i \), then

\[
\mu_A(u_i) = 1 - a_i, \quad \nu_A(u_i) = b_i, \quad \tau_A(u_i) = a_i - b_i
\]

From \( |\mu_B(u_i) - \mu_A(u_i)| = a_i \) and \( |\nu_B(u_i) - \nu_A(u_i)| = b_i \), it is

\[
\mu_B(u_i) = 1 - 2a_i, \quad \nu_B(u_i) = 2b_i, \quad \tau_B(u_i) = 2(a_i - b_i).
\]
Therefore,
\[ d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{(1 - a_i)(1 - 2a_i)} + \sqrt{2b_i^2 + 2(a_i - b_i)^2} \right) \]
\[ = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{2a_i} + \sqrt{(1 - a_i)(1 - 2a_i)} \right) \]
and
\[ d_s(A, E) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{1 - a_i} \right) \]

Obviously, we have
\[ \sqrt{2a_i} + \sqrt{(1 - a_i)(1 - 2a_i)} > \sqrt{1 - a_i} \]
Thus
\[ d_s(A, B) < d_s(A, E) \]
and dividing by \( n \)
\[ \frac{d_s(A, B)}{n} < \frac{d_s(A, E)}{n} \]

Lemma 3 shows that the extreme crisp sets with full memberships or full nonmemberships are categorically different from other intuitionistic fuzzy sets. With the same difference of memberships and nonmemberships, the distance from an extreme crisp set is always greater than the distances from other intuitionistic fuzzy sets. This conclusion agrees with our human perception about the quality change against quantity change and captures the semantic difference between extreme situation and intermediate situations.

5. SPHERICAL DISTANCES FOR FUZZY SETS

As we have already mentioned, fuzzy sets are particular cases of intuitionistic fuzzy sets. Therefore, the above spherical distances can be applied to fuzzy sets. In the following we provide Lemma 4 for the distance between two fuzzy sets.

**Lemma 4.** Let \( A = \{ (u_i, \mu_A(u_i)) : u_i \in U \} \) and \( B = \{ (u_i, \mu_B(u_i)) : u_i \in U \} \) be two fuzzy sets in the universe of discourse \( U = \{ u_1, u_2, \ldots, u_n \} \). Let \( a = \{ a_1, a_2, \ldots, a_n \} \) be a set of nonnegative real constants. If \( |\mu_A(u_i) - \mu_B(u_i)| = a_i \) holds for each
If \( u_i \in U \), then the following inequalities hold:

\[
\frac{2}{\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - a_i^2} \leq d_s(A, B) \leq \frac{2}{\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - a_i}
\]

\[
\frac{2}{n\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - a_i^2} \leq d_{ns}(A, B) \leq \frac{2}{n\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - a_i}
\]

The maximum distance between \( A \) and \( B \) is achieved if and only if one of them is a crisp set.

**Proof.** According to Definition 3, we have

\[
d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)\mu_B(u_i)} + \sqrt{v_A(u_i)v_B(u_i)} + \sqrt{\tau_A(u_i)\tau_B(u_i)} \right)
\]

For fuzzy sets, we have

\[
\tau_A(u_i) = 0 \quad \text{and} \quad v_A(u_i) = 1 - \mu_A(u_i)
\]

\[
\tau_B(u_i) = 0 \quad \text{and} \quad v_B(u_i) = 1 - \mu_B(u_i)
\]

Because \(|\mu_A(u_i) - \mu_B(u_i)| = a_i\) and \(\mu_A(u_i) \geq 0\), we have

\[
\mu_B(u_i) = \mu_A(u_i) \pm a_i
\]

If

\[
\mu_B(u_i) = \mu_A(u_i) + a_i
\]

then

\[
d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)\mu_B(u_i)} + \sqrt{(1 - \mu_A(u_i))(1 - \mu_B(u_i))} \right)
\]

\[
= \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)(\mu_A(u_i) + a_i)} \right.
\]

\[
+ \sqrt{(1 - \mu_A(u_i))(1 - \mu_A(u_i) - a_i)}
\]
This can be rewritten as

\[ ds(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos f_i(\mu_A(u_i)) \]

with \( f_i(t) = \sqrt{t(t + a_i)} + \sqrt{(1 - t)(1 - t - a_i)}, \ t \in [0, 1 - a_i]. \) The extremes of function \( f_i(t) \) will be among the solution of

\[ f_i'(t) = 0 \ t \in (0, 1 - a_i) \]

and the values 0 and 1 - \( a_i \), i.e., among \((1 - a_i)/2, 0 \) and \( 1 - a_i \). The maximum value \( \sqrt{1 - a_i^2} \) is obtained when \( t = (1 - a_i)/2 \), whereas the minimum value \( \sqrt{1 - a_i} \) is obtained in both 0 and \( 1 - a_i \). We conclude that

\[ \frac{2}{\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - a_i^2} \leq ds(A, B) \leq \frac{2}{\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - a_i} \]

When \( t = 0 \ or \ t = 1 - a_i \), we have, respectively, \( \mu_A(u_i) = 0 \) and \( \mu_B(u_i) = 1 - a_i + a_i = 1 \), which implies that one set among \( A \) and \( B \) has to be crisp in order to reach the maximum value under the given difference in their membership degrees.

Following a similar reasoning, it is easy to prove that the same conclusion is obtained in the case \( \mu_B(u_i) = \mu_A(u_i) - a_i \). If \( \mu_B(u_i) = \mu_A(u_i) - a_i \) for some \( i \), and \( \mu_B(u_j) = \mu_A(u_j) + a_j \) for some \( j \), then we could separate the elements into two different groups, each of them satisfying the inequalities, and therefore their summation obviously satisfying it too. The normalized inequality is obtained just by dividing the first one by \( n \).

Because spherical distances are quite different from the traditional distances, the semantics associated with them also differ. For the same relative difference in membership degrees, the spherical distance varies with the locations of its two relevant sets in the membership degree space, 2D for fuzzy sets and 3D for intuitionistic fuzzy sets. The spherical distance achieves its maximum when one of the fuzzy sets is an extreme crisp set. The following example illustrates this effect:

**Example 1.** Consider our previous example on human behavior. We can classify our behavior as perfect, good, acceptable, poor, and worst, with corresponding fuzzy membership of 1, 0.75, 0.5, 0.25, and 0, respectively. Let \( A = \{ (u, 0.75) : u \in U \} \), \( B = \{ (u, 0.5) : u \in U \} \), and \( E = \{ (u, 1) : u \in U \} \) be three fuzzy subsets, and \( U = \{ u \} \) is a universe of discourse with one element only.

From Section 3, we have

\[ d_1(A, B) = l_1(A, B) = e_1(A, B) = q_1(A, B) = 0.25 \]

\[ d_1(A, E) = l_1(A, E) = e_1(A, E) = q_1(A, E) = 0.25 \]
Obviously, we have
\[ d_1(A, B) = d_1(A, E), \quad l_1(A, B) = l_1(A, E) \]
\[ e_1(A, B) = e_1(A, E), \quad q_1(A, B) = q_1(A, E) \]

From Definition 3, we have
\[ d_{ns}(A, B) = d_s(A, B) \]
\[ = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{0.75 \ast 0.5 + (1 - 0.75)(1 - 0.5)} \right) = 0.17 \]
\[ d_{ns}(A, E) = d_s(A, E) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{0.75 \ast 1} \right) = 0.55 \]

We note that the traditional linear distance of fuzzy sets does not differentiate the semantic difference of a crisp set from a fuzzy set. However, \( d_s(A, E) \) and \( d_{ns}(A, E) \) are much greater than \( d_s(A, B) \) and \( d_{ns}(A, B) \). This demonstrates that the crisp set \( E \) is much more different from \( A \) than \( B \) although their membership difference are the same. Hence, the proposed spherical distance does show the semantic difference between a crisp set and a fuzzy set. This is useful when this kind of semantic difference is significant in the consideration.

Figure 3 shows four comparisons between the spherical distance and the Hamming distance for two fuzzy subsets \( A, B \) with a universe of discourse with one element \( U = \{u\} \). The curves represent the spherical distance, and straight lines denote Hamming distances. Figure 3a displays how the distance changes with respect to \( \mu_B(x) \) when \( \mu_A(x) = 0 \); Figure 3b uses the value \( \mu_A(x) = 1 \); in Figure 3c the value \( \mu_A(x) = 0.2 \) is used; finally \( \mu_A(x) = 0.5 \) is used in Figure 3d. Clearly, the spherical distance changes sharply for values close to the two lower and upper memberships values, but slightly for values close to the middle membership value. In the case of the Hamming distance, the same rate of change is always obtained.

Figure 4 displays how the spherical distance and Hamming distance changes with respect to \( \mu_B(x) \) for all possible values of \( \mu_A(x) \). Figure 4a shows that the spherical distance forms a curve surface, whereas a plane surface produced by the Hamming distance is shown in Figure 4b. Their contours in the bottom clearly show their differences. The contours for spherical distances are ellipses coming from \((0, 0)\) and \((1, 1)\) with curvatures increasing sharply near \((0, 1)\) and \((1, 0)\). Compared with these ellipses, the contours of Hamming distance are a set of parallel lines. These figures prove our conclusions in Lemma 4: The spherical distances do not remain constant as Hamming distances do when both sets experiment a same change in their membership degrees.
6. CONCLUSIONS

An important issue related with the representation of intuitionistic fuzzy sets is that of measuring distances. Most existing distances are based on linear plane
representation of intuitionistic fuzzy sets, and therefore are also linear in nature, in
the sense of being based on the relative difference between membership degrees.
In this paper, we have looked at the issue of 3D representation of intuitionistic
fuzzy sets. We have introduced a new spherical representation, which allowed us
to define a new distance function between intuitionistic fuzzy sets: the spherical
distance. We have shown that the spherical distance is different from those existing
distances in that it is nonlinear with respect to the change of the corresponding fuzzy
membership degrees, and thus it seems more appropriate than usual linear distances
for non linear contexts in 3D spaces.

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405.
According to Definition 3, we have
\[ d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)\mu_B(u_i)} + \sqrt{\nu_A(u_i)\nu_B(u_i)} + \sqrt{\tau_A(u_i)\tau_B(u_i)} \right) \]

Because \( A \) and \( B \) satisfy
\[ \mu_B(u_i) = \mu_A(u_i) + a_i \]
\[ \nu_B(u_i) = \nu_A(u_i) + b_i \]
then
\[ \tau_B(u_i) = 1 - \mu_A(u_i) - \nu_A(u_i) - a_i - b_i \]
and
\[ d_s(A, B) = \frac{2}{\pi} \sum_{i=1}^{n} \arccos \left( \sqrt{\mu_A(u_i)(\mu_A(u_i) + a_i)} + \sqrt{\nu_A(u_i)\nu_A(u_i) + b_i} \right) \]
\[ + \sqrt{(1 - \mu_A(u_i) - \nu_A(u_i))(1 - \mu_A(u_i) - \nu_A(u_i) - a_i - b_i)} \]

Let
\[ f(u, v) = \sqrt{u(u + a_i)} + \sqrt{v(v + b_i)} + \sqrt{(1 - u - v)(1 - u - v - a_i - b_i)}. \]

Denoting \( a = |a_i| \) and \( b = |b_i| \), then we distinguish 4 possible cases:
\[ \text{Case 1: } a_i \geq 0 \text{ and } b_i \geq 0. \text{ In this case,} \]
\[ f(u, v) = \sqrt{u(u + a)} + \sqrt{v(v + b)} + \sqrt{(1 - u - v)(1 - u - v - a - b)}. \]
Let \( f'(u, v)|_u = 0 \), we have
\[
u = \frac{a(v - 1)}{b} \quad \text{and} \quad v = \frac{a(1 - a - b - v)}{2a + b}.
\]

Let \( f'(u, v)|_v = 0 \), then
\[
v = \frac{b(u - 1)}{a} \quad \text{and} \quad v = \frac{b(1 - a - b - u)}{2b + a}.
\]

Because
\[
u = \frac{a(v - 1)}{b} < 0 \quad \text{and} \quad v = \frac{b(u - 1)}{a} < 0,
\]
the valid solutions are
\[
u = \frac{a(1 - a - b - v)}{2a + b} \quad \text{and} \quad v = \frac{b(1 - a - b - u)}{2b + a}.
\]
Solving these equations, we have
\[
u = \frac{a(1 - a - b)}{2(a + b)} \quad \text{and} \quad v = \frac{b(1 - a - b)}{2(a + b)},
\]
and therefore
\[
f_0 = f \left( \frac{a(1 - a - b)}{2(a + b)}, \frac{b(1 - a - b)}{2(a + b)} \right) = \sqrt{1 - (a + b)^2}.
\]
Obviously, the third square root in \( f(u, v) \) must be defined so
\[0 \leq u \leq 1 - a - b \quad \text{and} \quad 0 \leq v \leq 1 - a - b.
\]

The boundary points are reached when \( u \) and \( v \) get their minimum or maximum values. Denoting \( t = \tau_A(u_i) \), the following needs to be verified
\[u + v + t = 1.
\]

If \( u = v = 1 - a - b \) then \( t = 2a + 2b - 1 \), and \((1 - u - v)(1 - u - v - a - b) = (2a + 2b - 1)(a + b - 1) \leq 0 \), which implies that the third square root in \( f(u, v) \) is not defined. Therefore, we have three boundary points for \( A \)
\[
u = 0, \quad v = 1 - a - b, \quad t = a + b
\]
\[
u = 1 - a - b, \quad v = 0, \quad t = a + b
\]
and therefore
\[ f_1 = f(0, 1 - a - b) = \sqrt{(1 - a)(1 - a - b)} \]
\[ f_2 = f(1 - a - b, 0) = \sqrt{(1 - b)(1 - a - b)} \]
\[ f_3 = f(0, 0) = \sqrt{1 - a - b} . \]

If \( a_i \geq 0 \) and \( b_i \geq 0 \), then \( a + b \leq 1 \), and we have
\[ f_1 \leq f_3 \leq f_0 \]
\[ f_2 \leq f_3 \leq f_0 . \]

The relationship between \( f_1 \) and \( f_2 \) depends on the relationship between \( a \) and \( b \).

Let \( c = \max\{a, b\} \) and \( e = \min\{a, b\} \), then we have
\[ \sqrt{(1 - c)(1 - a - b)} \leq f(u, v) \leq \sqrt{1 - e^2} . \]

Case 2: \( a_i \leq 0 \) and \( b_i \leq 0 \). In this case, following a similar reasoning to the above one, the same conclusion is obtained.

Case 3: \( a_i \leq 0 \) and \( b_i \geq 0 \). In this case
\[ f(u, v) = \sqrt{u(u - a)} + \sqrt{v(v + b)} + \sqrt{(1 - u - v)(1 - u - v + a - b)} . \]

Let \( f'(u, v)|_{u} = 0 \), then
\[ u = \frac{a(1 - v)}{b} \text{ and } v = \frac{a(1 + a - b - v)}{2a - b} . \]

Let \( f'(u, v)|_{v} = 0 \), then
\[ v = \frac{b(1 - u)}{a} \text{ and } v = \frac{-b(1 + a - b - u)}{a - 2b} . \]

For \( u = a(1 - v)/b \) and \( v = b(1 - u)/a \), we get \( a = b \), hence \( u + v = 1 \), and then both \( A \) and \( B \) are fuzzy sets. According to Lemma 4, we know that
\[ \sqrt{1 - a} \leq f(u, v) \leq \sqrt{1 - a^2} . \]

Note that the following also holds: \((1 - a)(1 - a - b) \leq (1 - a)\).
For \( u = \frac{a(1 + a - b - v)}{2(a - b)} \) and \( v = \frac{-b(1 + a - b - u)}{a - 2b} \), we have

\[
u = \frac{a(1 + a - b)}{2(a - b)} \quad \text{and} \quad v = \frac{-b(1 + a - b)}{2(a - b)}.
\]

If \( a > b \) then \( u \geq 0 \) and \( v \leq 0 \), and therefore the only possibility here is \( v = 0 \), i.e. \( b = 0 \) in which case it is

\[
f_0 = f \left( \frac{1 + a}{2}, 0 \right) = \sqrt{1 - a^2}.
\]

If \( a < b \) then \( u \leq 0 \) and \( v \geq 0 \), and therefore \( a = 0 \), in which case it is

\[
f_1 = f \left( 0, \frac{1 - b}{2} \right) = \sqrt{1 - b^2}.
\]

For \( u = a(1 - v)/b \) and \( v = \frac{-b(1 + a - b - u)}{a - 2b} \), we have

\[
u = \frac{a(1 + b)}{2b} \quad \text{and} \quad v = \frac{1 - b}{2}
\]

and we obtain

\[
f \left( \frac{a(1 + b)}{2b}, \frac{1 - b}{2} \right) = \sqrt{1 - b^2} = f_1.
\]

Also, because \( u + v \leq 1 \) we have that \( (a - b) \cdot (1 + b) \leq 0 \) and, therefore, it is \( a \leq b \).

For \( u = \frac{a(1 + a - b - v)}{2a - b} \) and \( v = b(1 - u)/a \), we have

\[
u = \frac{1 + a}{2} \quad \text{and} \quad v = \frac{b(1 - a)}{2a}
\]

and

\[
f \left( \frac{1 + a}{2}, \frac{b(1 - a)}{2a} \right) = \sqrt{1 - a^2} = f_0.
\]

In this case it is \( a \geq b \).

The ranges for \( u \) and \( v \) are

\[
a \leq u \leq 1 \quad \text{and} \quad 0 \leq v \leq 1 - \max\{a, b\}
\]

therefore, we have the following boundary points for \( A \)

\[
u = a, \quad v = 1 - a, \quad t = 0
\]
\[ u = a, \quad v = 1 - b, \quad t = b - a \]
\[ u = 1, \quad v = 0, \quad t = 0 \]
\[ u = a, \quad v = 0, \quad t = 1 - a \]

Thus

\[ f_2 = f(a, 1 - a) = \sqrt{(1 - a)(1 - a + b)} \]
\[ f_3 = f(a, 1 - b) = \sqrt{1 - b} \]
\[ f_4 = f(1, 0) = \sqrt{1 - a} \]
\[ f_5 = f(a, 0) = \sqrt{(1 - a)(1 - b)} \]

The following inequalities hold

\[ f_2 \leq f_4 \leq f_0 \]
\[ f_5 \leq f_4 \leq f_0 \]
\[ f_5 \leq f_3 \leq f_1 \]

Denoting

\[ f_{\text{min}} = \sqrt{(1 - c)(1 - a - b)} \quad \text{and} \quad f_{\text{max}} = \sqrt{1 - e^2} \]

with \( e = \min\{a, b\} \) and \( c = \max\{a, b\} \), then we have

\[ \sqrt{1 - e^2} \geq f(u, v) \geq \sqrt{(1 - c)(1 - a - b)}. \]

**Case 4:** \( a_i \geq 0 \) and \( b_i \leq 0 \). In this case, following a similar reasoning to the above one, the same conclusion is obtained.

In the four cases, we conclude that

\[ \frac{2}{\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - e_i^2} \leq d_s(A, B) \leq \frac{2}{\pi} \sum_{i=1}^{n} \arccos \sqrt{(1 - c_i)(1 - |a_i| - |b_i|)} \]
\[ \frac{2}{n\pi} \sum_{i=1}^{n} \arccos \sqrt{1 - e_i^2} \leq d_s(A, B) \leq \frac{2}{n\pi} \sum_{i=1}^{n} \arccos \sqrt{(1 - c_i)(1 - |a_i| - |b_i|)} \]

where \( c_i = \max\{|a_i|, |b_i|\} \) and \( e_i = \min\{|a_i|, |b_i|\} \).
According to the boundary points discussed in the above four cases, if $a_i b_i \geq 0$ holds for each $u_i$, we have
\[ \mu_A(u_i) = 0, \quad \nu_A(u_i) = 1 - a - b, \quad \tau_A(u_i) = a + b \]
or
\[ \mu_A(u_i) = 1 - a - b, \quad \nu_A(u_i) = 0, \quad \tau_A(u_i) = a + b \]
hence, set $B$ has to satisfy
\[ \mu_B(u_i) = a, \quad \nu_B(u_i) = 1 - a, \quad \tau_B(u_i) = 0 \]
or
\[ \mu_B(u_i) = 1 - b, \quad \nu_B(u_i) = b, \quad \tau_B(u_i) = 0 \]
Obviously, $B$ is a fuzzy set. A similar conclusion can be drawn if $a_i b_i < 0$ holds for each $u_i$ and $\sqrt{(1 - c)(1 - c + e)} \leq \sqrt{(1 - a)(1 - b)}$. However, if $a_i < 0$ and $b_i > 0$ hold for each $u_i$ and $\sqrt{(1 - a)(1 - a + b)} > \sqrt{(1 - a)(1 - b)}$, we have
\[ \mu_A(u_i) = a, \quad \nu_A(u_i) = 0, \quad \tau_A(u_i) = 1 - a \]
hence, the set $B$ satisfies
\[ \mu_B(u_i) = 0, \quad \nu_B(u_i) = b, \quad \tau_B(u_i) = 1 - b \]
Obviously, $A$ is an intuitionistic fuzzy set with available information supporting only membership, whereas $B$ is an intuitionistic fuzzy set with available information supporting only nonmembership.