Consistency of 2D and 3D distances of intuitionistic fuzzy sets

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A B S T R A C T

For intuitionistic fuzzy sets, it is argued that three dimensional distance functions are not necessary because two dimensional distance functions already provide a simple and concise expression of the distance between two intuitionistic fuzzy sets. However, we show that a three dimensional interpretation of intuitionistic fuzzy sets could give different comparison results to the ones obtained with their two dimensional counterparts. In addition to the existing distances, we define a three dimensional Hausdorff distance and compare its consistency with its two dimensional counterpart, which shows the usefulness of the three dimensional functions to model and provide the expression of the distance between two intuitionistic fuzzy sets.

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1. Introduction

In fuzzy set theory, the comparison of sets is usually done by using a similarity or distance measure between their associated memberships and/or non-membership functions. For instance, for fuzzy sets just one parameter, the membership value, is needed and the distance between two fuzzy sets is usually defined as an aggregated value of the distances between the membership values of all their elements (Kacprzyk, 1997). The well known Hamming and Euclidean distances are among the generally agreed functions to be used to measure the distance between two fuzzy sets. For these functions, it is easy to prove that taking into account the non-membership value when calculating the distance between two fuzzy sets leads to equivalent results. The reason for the above is that the non-membership parameter is always known given the membership parameter due to the reciprocal additive property they comply. However, in the case of intuitionistic fuzzy sets, each element of that set has associated not two values (membership, non-membership) but three (hesitation). Therefore, with intuitionistic fuzzy sets, more parameters have to be taken into consideration when measuring their distance. Atanassov considers the distances between intuitionistic fuzzy sets as two dimensional functions and as such he defined the two dimensional (2D) distances for intuitionistic fuzzy sets (Atanassov, 1999). It is claimed in a parallel reasoning to the fuzzy sets case that the third parameter (hesitation) is always known given the other two (membership, non-membership).

Szmidt and Kacprzyk, however, proposed and applied three dimensional (3D) distances for intuitionistic fuzzy sets (Szmidt & Kacprzyk, 2000, 2001, 2003). Although the 3D distances are accepted as a correct representation model, many researchers do not consider them really necessary. Deschrijver, Cornelis, and Kerre (2004) proved that the 2D Euclidean and Hamming distances generate the same topology than their 3D counterparts given by Szmidt and Kacprzyk. Therefore, the mainstream opinion about 3D distances is that they are useless and not necessary and that hesitation margins are redundant information when measuring distances between intuitionistic fuzzy sets.

Distance is a measure of the similarity or difference between sets (Hausdorff, 1962). Therefore, for a selected distance measure, the relative order or ‘spatial distribution’ of sets is fixed with respect to a selected reference set. This is very important in real world application like information retrieval in databases, fuzzy number ranking (Trana & Duckstein, 2002), decision making and case based reasoning (Chen, 2011; Fang Zhang & Yang Liu, 2011; Li, 2005; Li, 2011; Szmidt & Baldwin, 2003, 2004, 2005; Szmidt & Kacprzyk, 1996a, 1996b, 1998, 2000b, 2001a, 2002a, 2002b, 2002c; Tan & Zhang, 2005; Wei, 2011; Xu, 2010; Yue, 2011). The information retrieval could be conducted consistently only when we have fixed ‘spatial distribution’, and a query like ‘find the 5 most similar sets with respect to set A’ could be implemented using a distance measure. If the hesitation margin was really redundant and the 3D distances could be completely replaced by the 2D distances, then the measuring results within the ‘spatial distribution’ of the measured sets have to match each other in both 2D and 3D representation, in other words, they should be consistent.

In this paper, based on a proposed notion of distance consistency, we show that the 2D interpretation does not always provide equivalent distance values between intuitionistic fuzzy sets to the ones that result by applying their 3D counterpart. In addition to the existing 3D distances, we define a 3D Hausdorff distance for intuitionistic fuzzy sets and compare it with its 2D counterpart.

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Interval-valued fuzzy sets (Sambuc, 1975) and intuitionistic fuzzy sets (Atanassov, 1986) are mathematically equivalent (Burillo & Bustince, 1996; Bustince & Burillo, 1996; Deschrijver & Kerre, 2003; Dubois, Ostaszewicz, & Prade, 2000; Wang & He, 2000). However, there is much debate in the fuzzy logic research community on the semantic differences between them (Cornelis, Atanassov, & Kerre, 2003). Due to mathematical equivalence, this paper focuses only on intuitionistic fuzzy sets and the result can be easily applied to interval-valued fuzzy sets as well.

The rest of the paper is set out as follows: Section 2 is devoted to the preliminaries and concepts and the existing 2D and 3D distances between intuitionistic fuzzy sets. Section 3 introduces the concept of ‘distance consistency’ and establishes the consistency and inconsistency between the 2D distances and their 3D counterparts when applied to fuzzy sets and to intuitionistic fuzzy sets. Then in Section 4, the extended Hausdorff distance is defined and its properties are given. In Section 5 we draw our conclusions.

2. Intuitionistic fuzzy sets and distances

Intuitionistic fuzzy sets were introduced by Atanassov (1986):

**Definition 1 (Intuitionistic fuzzy sets).** An intuitionistic fuzzy set $A$ in $X$ is given by

$$A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \}$$

where

$$\mu_A : X \rightarrow [0, 1], \quad \nu_A : X \rightarrow [0, 1]$$

and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in X.$$ 

For each $x$, the numbers $\mu_A(x)$ and $\nu_A(x)$ are the degree of membership and degree of non-membership of $x$ to $A$ respectively.

Obviously, an intuitionistic fuzzy set becomes a fuzzy set when $\nu_A(x) = 1 - \mu_A(x)$. However, when $\nu_A(x) \neq 1 - \mu_A(x)$, an extra parameter has to be taken into account when working with intuitionistic fuzzy sets: the hesitancy degree $\tau_A(x)$ of $x$ to $A$ (Atanassov, 1986, 1989, 1999)

$$\tau_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

The hesitancy degree $\tau_A(x)$ is an indicator of the hesitation margin of the membership of element $x$ to the intuitionistic fuzzy set $A$. It represents the amount of lacking information in determining the membership of $x$ to $A$.

2.1. Distances between fuzzy sets

The following functions have been proposed to measure the distance between two fuzzy sets $A$ and $B$ (Kacprzyk, 1997):

- The Hamming distance $d_1(A, B)$

$$d_1(A, B) = \frac{1}{n} \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) |$$

- The normalised Hamming distance $l_1(A, B)$

$$l_1(A, B) = \frac{1}{n} \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) |$$

- The Euclidean distance $e_1(A, B)$

$$e_1(A, B) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}$$

- The normalised Euclidean distance $q_1(A, B)$

$$q_1(A, B) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}$$

Two different approaches to measure the distances between intuitionistic fuzzy sets can be adopted. If only $\mu_0(x)$ and $\nu_0(x)$ are considered then a 2D distance is used as proposed by Atanassov in Atanassov (1999). However, if the third parameter, i.e. the hesitancy degree, is taken into account then a 3D distance is used as proposed by Szmidt and Kacprzyk (2000).

2.2. 2D Distances between Intuitionistic Fuzzy Sets

For two intuitionistic fuzzy subsets $A$ and $B$ defined on a finite universe of discourse $X$, Atanassov defined the distance functions between two intuitionistic fuzzy sets in Atanassov (1999) as:

- The Hamming distance $d_2(A, B)$

$$d_2(A, B) = \frac{1}{n} \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) | + | \nu_A(x_i) - \nu_B(x_i) |$$

- The normalised Hamming distance $l_2(A, B)$

$$l_2(A, B) = \frac{1}{2n} \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) | + | \nu_A(x_i) - \nu_B(x_i) |$$

- The Euclidean distance $e_2(A, B)$

$$e_2(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2$$

- The normalised Euclidean distance $q_2(A, B)$

$$q_2(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2$$

2.3. 3D distances between intuitionistic fuzzy sets

Szmidt and Kacprzyk (2000) modified the above distances to include the third parameter $\tau_A(x)$ as follows:

- The Hamming distance $d_3(A, B)$

$$d_3(A, B) = \frac{1}{n} \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) | + | \nu_A(x_i) - \nu_B(x_i) | + | \tau_A(x_i) - \tau_B(x_i) |$$

- The normalised Hamming distance $l_3(A, B)$

$$l_3(A, B) = \frac{1}{2n} \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) | + | \nu_A(x_i) - \nu_B(x_i) | + | \tau_A(x_i) - \tau_B(x_i) |$$

- The Euclidean distance $e_3(A, B)$

$$e_3(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\tau_A(x_i) - \tau_B(x_i))^2$$

- The normalised Euclidean distance $q_3(A, B)$

$$q_3(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\tau_A(x_i) - \tau_B(x_i))^2$$

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The aforementioned distances do not cover all possible distances defined for intuitionistic fuzzy sets, but they reflect the typical difference between 2D and 3D distance or similarity measures (Sheng Zhang, Jiang, Jia, & Luo, 2010; Wang & Xin, 2005; Zeng & Guo, 2008) of intuitionistic fuzzy sets. Therefore, we focus on these distances in this paper.

3. Comparison of 2D and 3D distances

Because the hesitant degree can be expressed in terms of the membership and non-membership degrees, it is argued that 3D distances are not necessary as their 2D counterparts provide sufficient measures. As we will show, this is not the case, because 2D and 3D functions could lead to contradictory results. Therefore, we claim that 3D geometrical representation of intuitionistic fuzzy sets can not be simply replaced by their 2D counterparts.

To do this, in the following, the concept of consistency of distances of intuitionistic fuzzy sets is introduced. As aforementioned, the distance between sets is a measure of the similarity or difference between these sets. Therefore, two distance measures should give consistent results if one could be replaced by the other.

Definition 2 (Consistent distances). For any three intuitionistic fuzzy subsets \( A, B, C \) of the universe of discourse \( X \) the distances \( d_1 \) and \( d_2 \) defined on \( X \) are said to be consistent if the following conditions hold:

1. \( d_1(A, C) = d_1(A, B) \Rightarrow d_2(A, C) = d_2(A, B) \)
2. \( d_1(A, C) > d_1(A, B) \Rightarrow d_2(A, C) > d_2(A, B) \)

Clearly, two distances for intuitionistic fuzzy sets which are consistent maintain the same order between any triple of intuitionistic fuzzy sets. Therefore, when two distances are consistent then one of them can be replaced with the other, with the only effect on the magnitude of the distances but no change on the order between intuitionistic fuzzy sets. It is easy to prove that the 2D (3D) Hamming distance is consistent with the 2D (3D) normalised Hamming distance. In particular, we have the following lemma for the 2D and 3D distances for fuzzy sets.

Lemma 1. The 2D distances in Eqs. (5)–(8) and 3D distances in (9)–(12) coincide with their 1D distances counterparts in (1)–(4) when the two sets in comparison are fuzzy sets.

Proof. We will prove the result just for the Hamming distances \( d_1, d_2, d_3 \) because in similar way, we can prove it for the other distances. For fuzzy sets \( A, B, C \) in \( U \), we have \( v_3(x) = 1 - \mu(x), v_4(x) = 1 - \mu(x) \) and \( \tau(x) = \tau(x) = 0 \). Thus

\[
d_2(A, B) = \frac{1}{2} \sum_{i=1}^{n} |[\mu_i(x) - \mu_i(x)] + |v_i(x) - v_i(x)| |
\]

\[
= \sum_{i=1}^{n} |\mu_i(x) - \mu_i(x)| = d_1(A, B)
\]

and

\[
d_3(A, B) = \frac{1}{2} \sum_{i=1}^{n} |\mu_i(x) - \mu_i(x)| + |v_i(x) - v_i(x)| + |\tau_i(x) - \tau_i(x)| |
\]

\[
= \sum_{i=1}^{n} |\mu_i(x) - \mu_i(x)| = d_1(A, B)
\]

Therefore, we have

\[
d_2(A, B) = d_3(A, B) = d_1(A, B)
\]

Obviously, 2D and 3D distance representations are really redundant for fuzzy sets because they provide the same results than their 1D distance counterparts. Clearly, for fuzzy sets because the distances in Eqs. (5)–(12) and (1)–(4) coincide with their counterparts we have that they are consistent.

Corollary 1. The 2D distances in Eqs. (5)–(8) and 3D distances in (9)–(12) are consistent to their 1D counterparts in (1)–(4) for fuzzy sets.

As aforementioned, it was argued that 3D distances on intuitionistic fuzzy sets were not necessary because their third parameter can be expressed in terms of the other two, and therefore the same results regarding the ordering of intuitionistic fuzzy sets would be obtained using their 2D counterparts. However, this is not the case as we show in the following:

Lemma 2. The 2D distances for intuitionistic fuzzy sets in Eqs. (5)–(8) are not consistent with the 3D distances in Eqs. (9)–(12).

Proof. We provide the proof just for the Euclidean distance, the proof for the rest being similar.

Let \( A = \{(x, 1 - 2v, v)\}, B = \{(x, v, 1 - 2v)\} \) and \( C = \{(x, v, v)\} \) be three intuitionistic fuzzy sets of the universe of discourse \( X = \{x\} \) with \( v \in [0, 0.5] \). According to Eqs. (7) and (11), we have

\[
e_1(A, B) = |1 - 3v|, \quad e_3(A, C) = |1 - 3v|
\]

\[
e_2(A, B) = |1 - 3v|, \quad e_3(A, C) = \sqrt{2}|1 - 3v|
\]

Therefore, we have \( e_2(A, B) > e_3(A, C) \) when \( e_2(A, B) = e_3(A, C) \), which obviously imply that \( e_2 \) and \( e_3 \) are not consistent. □

As the above result shows, the application of a 2D and a 3D distance to the same set of three intuitionistic fuzzy sets provides a different ordering or representation of it. Using the 3D distance both \( B \) and \( C \) are at the same distance from \( A \), while with the 2D distance \( B \) is further from \( A \) than \( C \). Clearly, this last result is due to the fact that the hesitation margins of the intuitionistic fuzzy sets are not taken into account. Although the hesitation margin can be derived from the other two, this does not mean that it has not an effect on the representation of the intuitionistic fuzzy sets.

For the above example, the difference in the results obtained can be seen clearly when comparing both the 2D and 3D geometrical representation of the three intuitionistic fuzzy sets. The three intuitionistic fuzzy sets \( A, B \) and \( C \) are represented as points \( A, B \) and \( C \) in 3D interpretation and \( A_2, B_2, C_2 \) in 2D interpretation, as shown in Fig. 1.

In Fig. 1, \( A_2B_2 \) is parallel to \( AB \), hence its length is not changed. However, \( A_2C_2 \) has an angle with \( AC \), and it is the projection of \( AC \) in plane \( \mu \nu \), therefore, its length is less than \( AC \). This is clearly a consequence of taking into account the third parameter of intuitionistic fuzzy sets. Although having the same relationship with the other two parameters in \( A, B, C \), the effect of taking it into account does not lead to the same results regarding their relative ordering.

Another argument to support the use of the three dimensions of intuitionistic fuzzy sets when calculating their distance is the following. If only two parameters among the three \((\mu(x), v(x), \tau(x))\) were sufficient to represent the distances because of their dependence \((\mu(x) + v(x) + \tau(x) = 1)\), then there would be no reason why the definition of the distance functions should based on \((\mu(x), v(x))\) and not on \((\mu(x), \tau(x))\) for example. As a consequence, the same results regarding the relative ordering should be obtained no matter which two parameters are used to measure the distance between intuitionistic fuzzy sets. Therefore, given three intuitionistic fuzzy sets their relative positions obtained with a 2D distance should be the same no matter we use \((\mu(x), v(x))\) or \((\mu(x), \tau(x))\). Again, this is not the case as
we show using the same intuitionistic fuzzy sets \( A, B \) and \( C \) of Lemma 2. Using the distance \( e_2 \) we get opposite conclusions:

\[ e_2(A, B) > e_2(A, C) \] for 2D interpretation based on \( \mu(x) \) and \( \nu(x) \)

\[ e_2(A, B) < e_2(A, C) \] for 2D interpretation based on \( \mu(x) \) and \( \tau(x) \)

From the point of view of his continuity or related concepts, the 3D distance does not reveal more than the 2D distance as proved in Deschrijver et al. (2004). However, the application of intuitionistic fuzzy sets may require more than a distinction between two sets. The same continuity does not guarantee the same order, and hence the result for a query may be different. The 3D distance reveals the impact of hesitation margins in the relative order. This is an important factor in decision making because it reflects the influence of lacking of information (Li, 2005; Szmidt & Baldwin, 2003, 2004, 2005; Szmidt & Kacprzyk, 1996a, 1996b, 1998, 2000b, 2001a, 2002a, 2002b, 2002c; Tan & Zhang, 2005). A query based on 2D distances may not reflect the same situation as if based on 3D distances. It is necessary to keep 3D distances as a supplement to the simplicity of 2D distances and therefore it is worthwhile to investigate the 3D Hausdorff distances for intuitionistic fuzzy sets.

### 4. Extended Hausdorff distances between intuitionistic fuzzy sets

Given two intervals \( U = [u_1, u_2] \) and \( V = [v_1, v_2] \) of \( \forall t \), the Hausdorff metric is defined as in Hung and Yang (2004)

\[ d_{h}(U, V) = \max\{|u_1 - v_1|, |u_2 - v_2|\} \]

The Hausdorff metric applied to two intuitionistic fuzzy sets, \( A(x) = [\mu_A(x), 1 - \nu_A(x)] \) and \( B(x) = [\mu_B(x), 1 - \nu_B(x)] \), gives the following:

\[ d_{h}(A(x), B(x)) = \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|\} \] (13)

The following 2D Hausdorff based distances between intuitionistic fuzzy sets have been proposed (Chen, 2007; Grzegorzewski, 2004; Hung & Yang, 2004).

- The Hamming distance \( d_{h}(A, B) \)

\[ d_{h}(A, B) = \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|, |\nu_i(A) - \nu_i(B)|\} \] (14)

- The normalised Hamming distance \( l_{h}(A, B) \)

\[ l_{h}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|, |\nu_i(A) - \nu_i(B)|\} \] (15)

- The Euclidean distance \( e_{h}(A, B) \)

\[ e_{h}(A, B) = \left( \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|^2, (\nu_i(A) - \nu_i(B))^2\} \right)^{1/2} \]

- The normalised Euclidean distance \( q_{h}(A, B) \)

\[ q_{h}(A, B) = \left( \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|^2, (\nu_i(A) - \nu_i(B))^2\} \right)^{1/2} \] (16)

- The normalised Hamming distance \( l_{h}(A, B) \)

\[ l_{h}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|, |\nu_i(A) - \nu_i(B)|\} \]

- The Euclidean distance \( e_{h}(A, B) \)

\[ e_{h}(A, B) = \left( \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|^2, (\nu_i(A) - \nu_i(B))^2\} \right)^{1/2} \] (18)

- The normalized Hamming distance \( l_{h}(A, B) \)

\[ l_{h}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|, |\nu_i(A) - \nu_i(B)|\} \] (19)

- The Euclidean distance \( e_{h}(A, B) \)

\[ e_{h}(A, B) = \left( \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|^2, (\nu_i(A) - \nu_i(B))^2\} \right)^{1/2} \] (20)

- The normalized Euclidean distance \( q_{h}(A, B) \)

\[ q_{h}(A, B) = \left( \sum_{i=1}^{n} \max\{|\mu_i(A) - \mu_i(B)|^2, (\nu_i(A) - \nu_i(B))^2\} \right)^{1/2} \] (21)

The following example shows that for the same set of intuitionistic fuzzy sets, opposite results can be derived when applying the above 3D extended Hausdorff distances and the corresponding 2D versions.

**Example 1.** Let us consider the following three intuitionistic fuzzy sets \( A, B, C \) in \( X = \{x\} \):

\[ A = \{(x, 0.25, 0.25)\}, \quad B = \{(x, 0.2, 0.2)\}, \quad C = \{(x, 0.18, 0.32)\} \]

The application of the 2D Hausdorff (14)–(17) results in...
Lemma 3. For intuitionistic fuzzy sets, the 2D Hausdorff distances are not consistent with the 3D extended Hausdorff distances.

In the following we present some properties of the 3D extended Hausdorff distances shared by the 2D Hausdorff distances.

Lemma 4. Let \( X \) denote a finite universe of discourse. All functions from Definition 3 are metrics.

Lemma 5. For any two intuitionistic fuzzy subsets \( A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \} \) and \( B = \{ (x, \mu_B(x), \nu_B(x)) : x \in X \} \) of the universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \), the following inequalities hold:

\[
d_{2D}(A, B) \leq d_3(A, B) \leq d_4(A, B), \quad I_2(A, B) \leq I_4(A, B) \leq I_4(A, B), \quad e_2(A, B) \leq e_4(A, B), \quad q_2(A, B) \leq q_4(A, B)
\]

Proof. For any nonnegative numbers \( a, b, c \), it is clear that max\(\{a, b, c\} \geq \max\{a, b, c\} \), hence \( d_2(A, B) \geq d_4(A, B) \). Obviously, \( \frac{1}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \leq \sum_{i=1}^{n} \max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\} \). Therefore, \( d_2(A, B) \leq d_4(A, B) \). Then we have \( d_2(A, B) \leq d_3(A, B) \leq d_4(A, B) \). The others inequalities can be proved in a similar way.

Lemma 6. For any two intuitionistic fuzzy subsets \( A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \} \) and \( B = \{ (x, \mu_B(x), \nu_B(x)) : x \in X \} \) of the universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \), the following inequalities hold:

\[
d_{\text{def}}(A, B) \leq \eta, \quad I_\text{def}(A, B) \leq \eta, \quad e_{\text{def}}(A, B) \leq \sqrt{\eta}, \quad q_{\text{def}}(A, B) \leq 1
\]

In contrast with conclusions from Grzegorzewski (2004), we note that these inequalities do not include those 3D representations in Eqs. (9)–(12). The reason for this being that Lemma 4 in Grzegorzewski (2004) does not hold for some special cases. For example, when \( A = (\{x_0, 0.7, 0.1\}, B = (\{x_0, 0.3, 0.2\}) \) then \( e_4(A, B) = 0.4 > e_6(A, B) = 0.36 \).

Lemma 5 presents a general relationship between 3D Hausdorff distances and 2D Hausdorff distances, but lacks to provide conditions to assure their consistency. The following results presents a condition under which both 2D Hausdorff distances and 3D extended Hausdorff distances are consistent, and therefore the same conclusions can be derived no matter which Hausdorff distance is used.

Lemma 7. Given any two intuitionistic fuzzy subsets \( A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \} \) and \( B = \{ (x, \mu_B(x), \nu_B(x)) : x \in X \} \) of the universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \), the following relationship holds between 3D extended Hausdorff distances and 2D Hausdorff distances: If \( (\mu_A(x_i) - \mu_B(x_i)) + \nu_A(x_i) - \nu_B(x_i) \leq 0 \) for each \( x_i \in X \), then \( d_3(A, B) = d_4(A, B) \), \( I_3(A, B) = I_4(A, B) \), \( e_3(A, B) = e_4(A, B) \), \( q_3(A, B) = q_4(A, B) \)

Proof. We only prove the results for the Hamming distance, because the other relationships can be proved in a similar way. When \( (\mu_A(x_i) - \mu_B(x_i)) + \nu_A(x_i) - \nu_B(x_i) \leq 0 \), \( \tau_A(x_i) - \tau_B(x_i) \leq \max\{\mu_A(x_i) - \mu_B(x_i), \nu_A(x_i) - \nu_B(x_i)\} \) holds. Therefore,

\[
\max\{\mu_A(x_i) - \mu_B(x_i), \nu_A(x_i) - \nu_B(x_i)\} = \max\{\mu_A(x_i) - \mu_B(x_i), \nu_A(x_i) - \nu_B(x_i)\} \quad \forall \ x_i \in X
\]

Hence, \( d_3(A, B) = d_4(A, B) \).

The following result expresses the relationship between the 3D distances in Eqs. (9)–(12) and the 3D extended Hausdorff distances.

Lemma 8. Given any two intuitionistic fuzzy subsets \( A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \} \) and \( B = \{ (x, \mu_B(x), \nu_B(x)) : x \in X \} \) of the universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \), the following relationships hold:

\[
d_{\text{def}}(A, B) = d_3(A, B), \quad I_\text{def}(A, B) = I_3(A, B), \quad e_{\text{def}}(A, B) = e_3(A, B), \quad q_{\text{def}}(A, B) = q_3(A, B)
\]

Proof. The same result can be obtained for \( I_\text{def} \) and \( d_\text{def} \). For the Euclidean distance, we get:

\[
\max\{\mu_A(x) - \mu_B(x)\} + \nu_A(x) - \nu_B(x) = \max\{\nu_A(x) - \nu_B(x)\} \text{ is equal to the maximum of } \{\mu_A(x) - \mu_B(x), \nu_A(x) - \nu_B(x)\} \text{ and } |\tau_A(x) - \tau_B(x)|, \text{ and therefore } d_3(A, B) = d_\text{def}(A, B).
\]

The same result can be obtained for \( q_3 \) and \( q_{\text{def}} \).

5. Conclusions

It is claimed that 3D distances do not perform better than their 2D counterparts, because the third parameter of intuitionistic fuzzy sets can be expressed in terms of the other two. However, we have shown that 2D distances are not consistent with their corresponding 3D distances, and the application of a 2D distance and a 3D distance to the same set of three intuitionistic fuzzy sets may provide a different ordering or representation of it. Clearly, the reason for this is that the third dimension has an effect on the representation of the intuitionistic fuzzy sets. We conclude that the use of the third parameter is significant, and that 2D distances cannot simply replace their 3D counterparts. We have introduced 3D
extended Hausdorff distances for intuitionistic fuzzy sets, and have presented some of their properties.

References


